PHILOSOPHICAL

TRANSACTIONS

OF THE

ROYAL SOCIETY

OF

LONDON.

FOR THE YEAR MDCCCLXXXI.

VOL. 172.—PART III.

LONDON:

PRINTED BY HARRISON AND SONS, ST. MARTIN'S LANE, W.C.,

Printers in Ordinary to Her Majesty.

MDCCCLXXXII.

CONTENTS.

PART III.

XII. *Researches in Spectrum-Analysis in connexion with the Spectrum of the Sun.*—*By* J. Norman Lockyer, *F.R.S.* . . . . . page 561

XIII. *Researches on the Minute Structure of the Thyroid Gland. By* E. Cresswell Baber, *M.B., Lond. Communicated by* E. Klein, *M.D., F.R.S.* 577

XIV. *On Toroidal Functions. By* W. M. Hicks, *M.A., Fellow of St. John's College, Cambridge. Communicated by* J. W. L. Glaisher, *M.A., F.R.S.* . . . 609

XV. Polacanthus Foxii, *a large undescribed Dinosaur from the Wealden Formation in the Isle of Wight. By* J. W. Hulke, *F.R.S.* 653

XVI. *On the Histology and Physiology of Pepsin-forming Glands. By* J. N. Langley, *M.A., Fellow of Trinity College, Cambridge. Communicated by Dr.* M. Foster, *F.R.S.* 663

XVII. *Memoir on* Abel's *Theorem. By* R. C. Rowe, *M.A., Fellow of Trinity College, Cambridge. Communicated by* A. Cayley, *LL.D., F.R.S., Sadlerian Professor of Mathematics in the University of Cambridge* 713

Addition to Mr. Rowe's *Memoir. By Professor* Cayley, *F.R.S.* 751

XVIII. *On* Riccati's *Equation and its Transformations, and on some Definite Integrals which satisfy them. By* J. W. L. Glaisher, *M.A., F.R.S., Fellow of Trinity College, Cambridge* 759

XIX. The Croonian Lecture.—*Observations on the Locomotor System of Echinodermata. By* George J. Romanes, *M.A., F.R.S., and Professor* J. Cossar Ewart, *M.D.* 829

XX. *On the Influence of the Molecular Grouping in Organic Bodies on their Absorption in the Infra-red Region of the Spectrum. By Captain* Abney, *R.E., F.R.S., and Lieut.-Colonel* Festing, *R.E.* 887

Index to Volume 919

27783

LIST OF ILLUSTRATIONS.

Plate 67.—Mr. J. Norman Lockyer on Spectrum Analysis in Connexion with the Spectrum of the Sun.

Plates 68, 69.—Mr. E. C. Baber on the Minute Structure of the Thyroid Gland.

Plates 70 to 76.—Mr. J. W. Hulke on the *Polacanthus Foxii*.

Plates 77, 78.—Mr. J. N. Langley on the Histology and Physiology of Pepsin-forming Glands.

Plates 79–85.—Mr. G. J. Romanes and Professor J. C. Ewart on the Locomotor System of Echinodermata.

Plates 86–88.—Captain Abney and Lieutenant-Colonel Festing on the Molecular Grouping of Organic Bodies.

FRANKLIN INSTITUTE

Adjudication of the Medals of the Royal Society for the year 1881, by the President and Council.

The Copley Medal to Prof. Karl Adolph Wurtz, For. Mem. R.S., for his discovery of the Organic Ammonias, the Glycols, and numerous other investigations which have exercised considerable influence on the Progress of Chemistry.

A Royal Medal to Mr. Francis Maitland Balfour, F.R.S., for his numerous and important contributions to Animal Morphology, and more especially for his investigations respecting the origin of the Uro-genital Organs and the Cerebro-spinal Nerves of the Vertebrata, and for his work on the Development of the Elasmobranch Fishes.

A Royal Medal to the Rev. John Hewitt Jellett for his various Mathematical and Physical Papers, more especially for his researches in Chemical Optics, and his invention of the new and delicate Analyzer by which they were carried out.

The Davy Medal to Prof. Adolf Baeyer for his Synthesis of Indigo.

The Bakerian Lecture "On the Action of Free Molecules on Radiant Heat, and its Conversion thereby into Sound," was delivered by Dr. Tyndall, F.R.S.

The Paper "Observations on the Locomotor System of Echinodermata," by Mr. George J. Romanes, M.A., F.R.S., and Prof. J. Cossar Ewart, M.D., was appointed as the Croonian Lecture.

FRANKLIN INSTITUTE LIBRARY

PHILOSOPHICAL

TRANSACTIONS

OF THE

ROYAL SOCIETY

OF

LONDON.

FOR THE YEAR MDCCCLXXXI.

VOL. 172.

LONDON:

PRINTED BY HARRISON AND SONS, ST. MARTIN'S LANE, W.C.,

Printers in Ordinary to Her Majesty.

MDCCCLXXXII.

CONTENTS

OF VOL. 172.

PART I.

I. *On the Structure and Development of the Skull in the Batrachia.*—Part III. *By* William Kitchen Parker, *F.R.S.* . . . . . page 1

PART II.

II. *The Cochlea of the* Ornithorhynchus platypus *compared with that of ordinary Mammals and of Birds. By* Urban Pritchard, *M.D., F.R.C.S., Aural Surgeon to King's College Hospital. Communicated by Professor* Huxley, *Sec.R.S.* 267

III. *On the Organization of the Fossil Plants of the Coal-Measures.*—Part XI. *By* W. C. Williamson, *F.R.S., Professor of Botany in the Victoria University (Owens College, Manchester)* 283

IV. *On the Induction of Electric Currents in Infinite Plates and Spherical Shells. By* C. Niven, *M.A., Professor of Natural Philosophy in the University of Aberdeen. Communicated by* J. W. L. Glaisher, *M.A., F.R.S.* . . . 307

V. *Electrostatic Capacity of Glass,* II., *and of Liquids. By* J. Hopkinson, *M.A., D.Sc., F.R.S.* 355

VI. *On the Forty-eight Coordinates of a Cubic Curve in Space. By* William Spottiswoode, *P.R.S.* 375

VII. *On the Viscosity of Gases at High Exhaustions.* By WILLIAM CROOKES, *F.R.S.* . . . . . page 387

Note on the Reduction of Mr. CROOKES's *Experiments on the Decrement of the Arc of Vibration of a Mica Plate oscillating within a Bulb containing more or less Rarefied Gas.* By *Professor* G. G. STOKES, *Sec.R.S.* 435

VIII. *On the Electrical Resistance of Thin Liquid Films, with a revision of* NEWTON'S *Table of Colours.* By A. W. REINOLD, *M.A., Professor of Physics in the Royal Naval College, Greenwich, and* A. W. RÜCKER, *M.A., Professor of Physics in the Yorkshire College, Leeds. Communicated by Professor* W. GRYLLS ADAMS, *F.R.S.* 447

IX. *On the Tidal Friction of a Planet attended by several Satellites, and on the Evolution of the Solar System.* By G. H. DARWIN, *F.R.S.* 491

X. *On the Thermal Conductivity of Water.* By J. T. BOTTOMLEY. *Communicated by Sir* WILLIAM THOMSON, *LL.D., F.R.S., Professor of Natural Philosophy in the University of Glasgow* 537

XI. *Description of some Remains of the Gigantic Land-Lizard* (Megalania prisca, OWEN), *from Australia.*—Part III. By *Professor* OWEN, *C.B., F.R.S., &c.* 547

Index to Part II. 557

PART III.

XII. *Researches in Spectrum-Analysis in connexion with the Spectrum of the Sun.*—No. V. By J. NORMAN LOCKYER, *F.R.S.* 561

XIII. *Researches on the Minute Structure of the Thyroid Gland.* By E. CRESSWELL BABER, *M.B. Lond. Communicated by* E. KLEIN, *M.D., F.R.S.* 577

XIV. *On Toroidal Functions.* By W. M. HICKS, *M.A., Fellow of St. John's College, Cambridge. Communicated by* J. W. L. GLAISHER, *M.A., F.R.S.* . . . 609

XV. Polacanthus Foxii, *a large undescribed Dinosaur from the Wealden Formation in the Isle of Wight.* By J. W. HULKE, *F.R.S.* 653

XVI. *On the Histology and Physiology of Pepsin-forming Glands.* By J. N. LANGLEY, *M.A., Fellow of Trinity College, Cambridge. Communicated by Dr.* M. FOSTER, *F.R.S.* 663

XVII. *Memoir on* ABEL'S *Theorem. By* R. C. ROWE, *M.A., Fellow of Trinity College, Cambridge. Communicated by* A. CAYLEY, *LL.D., F.R.S., Sadlerian Professor of Mathematics in the University of Cambridge* . . . . . page 713

Addition to Mr. ROWE'S *Memoir. By Professor* CAYLEY, *F.R.S.* . . . 751

XVIII. *On* RICCATI'S *Equation and its Transformations, and on some Definite Integrals which satisfy them. By* J. W. L. GLAISHER, *M.A., F.R.S., Fellow of Trinity College, Cambridge* 759

XIX. THE CROONIAN LECTURE.—*Observations on the Locomotor System of Echinodermata. By* GEORGE J. ROMANES, *M.A., F.R.S., and Professor* J. COSSAR EWART, *M.D.* 829

XX. *On the Influence of the Molecular Grouping in Organic Bodies on their Absorption in the Infra-red Region of the Spectrum. By Captain* ABNEY, *R.E., F.R.S., and Lieut.-Colonel* FESTING, *R.E.* 887

Index to Volume 919

FRA... INSTI... LIBRARY

LIST OF ILLUSTRATIONS.

Plates 1 to 44.—Mr. W. K. Parker on the Structure and Development of the Skull in the Batrachia.

Plates 45, 46.—Dr. U. Pritchard on the Cochlea of the *Ornithorhynchus platypus* compared with that of ordinary Mammals and of Birds.

Plates 47 to 54.—Professor W. C. Williamson on the Organization of the Fossil Plants of the Coal-Measures.

Plates 55 to 58.—Mr. W. Crookes on the Viscosity of Gases at High Exhaustions.

Plates 59, 60.—Professors A. W. Reinold and A. W. Rücker on the Electrical Resistance of Thin Liquid Films, with a revision of Newton's Table of Colours.

Plates 61 to 63.—Mr. G. H. Darwin on the Tidal Friction of a Planet attended by several Satellites, and on the Evolution of the Solar System.

Plates 64 to 66.—Professor Owen on some Remains of the Gigantic Land-Lizard (*Megalania prisca*) from Australia.

Plate 67.—Mr. J. Norman Lockyer on Spectrum Analysis in Connexion with the Spectrum of the Sun.

Plates 68, 69.—Mr. E. C. Baber on the Minute Structure of the Thyroid Gland.

Plates 70 to 76.—Mr. J. W. Hulke on the *Polacanthus Foxii*.

Plates 77, 78.—Mr. J. N. Langley on the Histology and Physiology of Pepsin-forming Glands.

Plates 79–85.—Mr. G. J. Romanes and Professor J. C. Ewart on the Locomotor System of Echinodermata.

Plates 86–88.—Captain Abney and Lieutenant-Colonel Festing on the Molecular Grouping of Organic Bodie

XII. *Researches in Spectrum-Analysis in connexion with the Spectrum of the Sun.*—No. V.

By J. Norman Lockyer, *F.R.S.*

Received April 29,—Read May 23, 1878.

[Plate 67.]

Contents.

	Page.
I. Introduction	561
II. Refraction Map	563
A. Instruments, &c.	563
B. Method of Mapping	563
a. Construction of Interpolation Curves	563
β. Construction of the Map	564
γ. Determination of Wave-lengths, new lines	565
III. Diffraction Map	566
A. Instruments, &c.	566
B. Determination of Wave-lengths	567
C. Construction of the Map	568
IV. Importance of the Photographic Method	569
V. Photographic Process Employed	571
VI. Conclusion	572
Tables.	
I. Table of Lines used in construction of Interpolation Curves (Refraction Map)	573
II. Table of Lines Interpolated from Curves (Refraction Map)	574
III. Table of Wave Lengths	575

FRANKLIN INSTITUTE LIBRARY

I. INTRODUCTION.

It has long been clear that the means placed at our disposal by photography for studying the solar spectrum enable us to construct maps of the region more refrangible than *b* on a much larger scale than those hitherto employed.

At the same time, as our knowledge of the molecular conditions under which changes in spectra occur is increased, it becomes necessary to embrace more and more detail in the inquiry.

Hence in former communications to the Royal Society I have pointed out that in order to increase our knowledge of the sun's chemical constitution, and to have a

ready and unfailing means of detecting cyclical changes, maps of the spectrum on a large scale must be constructed.

By means of the photographic method described in the third Memoir of this series,* and illustrated by its application to the mapping of the spectra of barium, calcium and strontium in the fourth,† I commenced in the year 1875 a new map of the solar spectrum on four times the scale of ÅNGSTRÖM's "Spectre Normal." Specimens of this map in its earlier stages were laid before the Society, with a preliminary note, in November of that year.‡ After the presentation of this preliminary note, the construction of the map was carried on, until on January 10, 1877, I submitted to the Royal Society a complete Memoir on the first part completed (W.L. 390–400 millionths millim.) with comparisons of the lines of 25 metals and complete tables, both of the solar and metallic lines.

While this Memoir was in the hands of the referees, I received from my friend Mr. L. M. RUTHERFURD, of New York, a magnificent reflection grating, with 17,280 lines to the inch. This enabled me to attack the question of wave-lengths in a much more satisfactory manner than I had been able to do in the first instance. I therefore determined to reject my two years' work and to do it all over again, in order, if possible, to introduce greater accuracy than the method of graphical interpolation, which I had been compelled to adopt in the first instance, had permitted. I therefore applied to the Royal Society for leave to withdraw my paper, giving the above-stated reason for so doing, and at once received the required permission.

I now beg to re-submit to the Society that part of the withdrawn Memoir which has reference to the spectrum of the sun, independently of the spectra of the metals. I have determined to do this, not only because the reviewing of the wave-lengths of the metallic lines will take considerable time, but because during the time that the Memoir was in the hands of the Royal Society, and that which has elapsed since I received it back, I have very nearly completed the survey of the whole of the metals for this region, so that the comparison can now shortly be given for the whole of the metallic elements united for this part.

The section of the new normal spectrum comprises, as I have said, the FRAUNHOFER lines between W.L. 390-400 millionths millim. That portion of the map now in question is but a very small fraction of the whole region of the spectrum workable by photography, and my chief object in thus forwarding a fragment to the Royal Society is to point out the necessity for, and to invite co-operation in, a work of such magnitude, and to publish full details of the methods I have found most effective for the use of those who may take up the research. For this reason, and because it may happen that a diffraction grating of the requisite brilliancy may not be forthcoming in all cases, I think it best to give a complete history of the production of the map, including that stage of graphical interpolation of wave-lengths which I have, by Mr. RUTHERFURD's generosity, been able to supersede.

* Phil. Trans., Vol. 164, Part II., p. 479. † Phil. Trans., Vol. 164, Part II., p. 805.
‡ Proc. Roy. Soc., 158, 1875.

II. REFRACTION MAP.

A. Instruments, &c.

The same general arrangement of the spectroscope, electric lamp, and lenses figured in the third of the present series of memoirs has been employed for obtaining the photographs used in the construction of the refraction map.

The spectroscope used is one constructed on the model of Bunsen and Kirchhoff, by Schmidt and Hensch, of Berlin. It is provided with a train of four flint glass prisms, viz., three of 45° and one of 60°.

The prisms are levelled by Kirchhoff's method, and adjusted as nearly as possible for the minimum deviation of the centre of the section being worked upon.

The camera employed is provided with a simple quartz lens of 5-feet focus. The image of the spectrum formed by this lens falls on a sensitized glass plate, 16·6 × 5·8 centims. The dispersion obtained by this arrangement gives a photographic impression, in which the distance from G to K (using Cornu's nomenclature) is about 11 centims. and from H to K about 1·3 centims.

The focus is first fixed approximately by viewing the image of the spectrum on the ground glass focussing screen by means of a positive eye piece, and the final adjustment is determined by a series of trial plates, using the solar spectrum when possible, or, in the absence of the sun, the spectrum of some metal such as cerium, which contains an immense number of lines throughout its entire length. For fine definition it is requisite that if the beam of light which falls on the slit is not parallel, the light of the wave-length to be photographed should be brought to focus on it.

B. Method of Mapping.

a. Construction of Interpolation Curves.

The relative positions of all the most prominent lines visible in an enlargement of the photographs employed were first laid down on a strip of paper and then transferred to the horizontal line ruled at the base of some curve paper to furnish the ordinates. The lines thus selected were referred to Cornu's map, and their wave-lengths furnished the abscissæ. A wave-length scale was marked off on the vertical line, 4 millims. of which represented 1 millim. of Cornu's map. In the case of lines assigned to particular metals by former observers, a photograph of the metallic and solar spectra confronted was found useful in this work of identification. The wave-lengths of the selected lines having been found, vertical lines are carried up from them, and horizontal lines marked across from the corresponding wave-lengths. Where the vertical and horizontal lines meet, a series of points is obtained, through which is drawn a curve with as much regularity as possible. The drawing of this curve at once reveals errors of identification or of wave-length in the map used.

Two curves (A and B) were employed in the construction of this section. The first curve (A) was obtained from an enlargement on glass, and was available from about W.L. 386·0 to 399·98 millionths millim. The distance between K and H in this photograph is 3·5 centims., and it was employed in preference to Rutherfurd's print, because being in much better focus it was more easy, in consequence, to compare it with Cornu's map.

The second curve (B) was constructed from Rutherfurd's print, commencing at the point where the photograph referred to in the last paragraph fails; this curve extends from W.L. 397·045 to W.L. 400·925 millionths millim.

The wave-lengths are expressed in x^{th} metres, and are given in two places of decimals. This has been found necessary in order to distinguish between lines very close together. The greatest error in the method of determining wave-lengths by graphical interpolation does not much exceed 2 millims. of the present scale $= \frac{1}{5}$ millim. Cornu's scale $= \frac{1}{20,000,000}$ millim. in the actual wave-length.

I have not thought it necessary to reproduce the curves, but at the end of the paper I give two tables which will show the accuracy which it is possible to secure by the method above described. The experience gained in constructing these curves (I had already completed the region W.L. 390–440 millionths millim. before I received the grating, of which mention has already been made, from Mr. Rutherfurd) leads to the following conclusions :—

I. The photograph of the solar spectrum employed should be on as large a scale as possible.

II. The wave-length scale giving the abscissæ should never be smaller than that used here.

III. The photograph should be tested for distortion after enlargement.

IV. The fundamental lines should be copied from the glass and not from an enlargement on paper.

β. Construction of the Map.

The photograph used for the construction of the original refraction map is the original negative varnished and examined by a lens, or placed under a simple microscope of low magnifying power. Considerable loss of detail and sharpness, leading to the obliteration of faint lines and the fusion together of close groups, has been found to be an invariable effect of photographic enlargements. Thus in section W.L. 390–400 millionths millim., there were mapped 333 lines in the refraction map as completed, whereas in the map of the same section exhibited to the Society in November, 1875, which was drawn from an enlargement, there were but 221 lines. I may add that the same holds good with metallic spectra. Thus the spectrum of cerium as first mapped from enlargements contained several broad and nebulous bands which the negatives now in use resolve into groups of distinct but closely-packed lines.

The drawing of section W.L. 390–400 millionths millim., exhibited with the preliminary note before referred to, was utilised for the first revision of the refraction map, for which purpose it was sufficiently good, although not good enough for copying finally, having been produced, as before mentioned, from a photographic enlargement of this region of the spectrum.

The method adopted in drawing the final refraction map was the following. A millimetre scale of the required length having been drawn near the top of a sheet of paper prepared for the reception of the map, the fundamental lines were first entered in their correct positions in pencil upon it, and afterwards made of their correct widths and intensities, as seen in the standard photograph, by means of a drawing pen and Indian ink. The width of these fundamental lines, where sufficiently great, was found by actual measurements on the curve. The spaces between these fundamental lines being too large on the scale adopted for interpolating the intermediate lines accurately by eye, their places in the photograph were found and their corresponding wave-lengths determined by means of the curve.

The lines obtained in this manner by the aid of the interpolation curves were entered in the map in the same manner as before.

A skeleton map is thus obtained, in which no greater extent is left blank than 8 or 10 millims.—a distance sufficiently small, considering the shape of the curve, to allow of the interpolation of the intermediate lines by eye. The standard photograph, placed under a microscope of low magnifying power, was accordingly made to furnish the details for the remaining lines, which were interpolated with the greatest possible care, the interpolation position being checked from time to time by actual measurements of the wave-lengths of the selected lines on the curve.

In all cases the agreement was as close as could be desired.

In order to save the cost of engraving, a third map was constructed from the finally revised copy on twelve times the scale of Ångström's. This enlarged map was photographed down to the original scale adopted for this work, viz., four times the scale of Ångström. For this enlarged map I was greatly indebted to Lance-Corporal Murray, R.E., who made the drawing with great skill and care.

γ. *Determination of the Wave-lengths of the Solar Lines.*

I have no means for determining wave-lengths absolutely, but the accurate measurement of the wave-length of the chief lines in the ultra-violet portion of the solar spectrum by Cornu rendered it unnecessary to attempt any absolute measurements for the construction of the present map. The last degree of accuracy in the determination of wave-lengths is also, I think, having regard to the objects I have had in view, of less importance than good maps showing the details of the spectrum. The idea was at first entertained of simply copying Cornu's map on the scale now adopted, and interpolating new lines by eye; but for two reasons this idea had to be abandoned.

In the first place, CORNU's map, although by far the best ever published of this region of the spectrum, was found to be considerably wanting in detail when compared with the best negatives taken with the camera before described. For example, the following lines are represented as single in CORNU's map:—

Section 3900–4000.

Approximate wave-length.	State in new map.
3920·4	Double.
3920·8	,,
3921·3	Triple.
3935·2	Double.
3937·8	,,
3952·0	Triple.
3965·0	Double.
3965·9	,,
3993·3	,,
3993·9	,,
3997·3	,,

In the next place, on the increased scale of the new map, the omission of lines by CORNU gives rise to blank spaces of too great an extent to enable one to interpolate new lines by eye with sufficient accuracy.

CORNU admits that he has attempted to give groups of lines a natural appearance (when contrasted with photography), without special reference to the position of lines within the groups. Slight errors thus introduced, although unimportant on ÅNGSTRÖM's scale, would be considerably exaggerated on the scale now adopted.

The table of wave-lengths of the refraction map depends then (1) upon CORNU's values as far as they are available, and (2) upon the interpolation curves for all the other lines. I have already stated the degree of accuracy which they may be expected to possess.

III. DIFFRACTION MAP.

A. INSTRUMENTS, &c.

The stand and collimator of the spectroscope of the BUNSEN and KIRCHHOFF model already referred to were utilised, the train of prisms being removed and replaced by the grating. This grating, which as I have already said I owe to the great kindness of Mr. RUTHERFURD, contains 13,321 lines 25 millims. long (17,280 to the inch). The ruled portion occupies the centre of a plate of glass 40 millims. square; a deposit of silver is made on the ruled side, and then another plate is cemented with Canada balsam to the silvered side, to protect the surface. The grating is mounted with

wood at the back, over the centre of a brass circular plate free to move on another lower plate levelled by screws.

The observing telescope, after a few preliminary observations to adjust the grating for verticality, was replaced by the long camera before mentioned.

The focus for any one order was determined in much the same way as in the case of the refraction photograph; that is, it was first roughly obtained by viewing the spectrum on the ground glass screen by means of a fixed positive eye piece, but the final adjustment was likewise determined by a series of trial plates, and, when satisfactory, the sliding end of the camera was clamped up tight in order to keep the adjustment as rigid as possible.

The spectrum of the second order was first obtained.

In this the distance between the H and K lines was nearly 8 millims.

With the angle between the collimator and the camera which I used, the H and K lines of the third order are brought to focus with the D lines of the second order, so that practically the grating and instruments generally are in the first instance adjusted for D, with the exception that the violet light is focussed on to the slit.

In the photograph thus obtained the H and K lines are about 15 millims. apart, and the perfection of the grating is demonstrated by the definition of the photographs, except in those parts of them where, in consequence of the presence of the glass plate, the purity of the spectrum is marred by interference effects.*

As the exposure for the third order spectrum was much longer than that required by the train of prisms, the photographs used were taken by means of a siderostat, the use of which gives great constancy to the direction of the beam of light employed. At times, an opera glass was used for obtaining a parallel beam; at others, the centre of the beam (of 12 inches diameter) thrown by the siderostat was grasped by an 8-inch object glass, and focussed on the slit.

B. Determination of Wave-lengths.

The small original negative was enlarged by many stages to obviate all chances of distortion, until a glass positive was obtained of sufficient size to commence operations upon. This was 24 centims. in length. The wave-lengths of the extreme points having been taken from Cornu's map, it was divided into ten parts carefully by compasses. It was then found that the intermediate wave-lengths did not correspond with those of Cornu. I at first feared that some distortion had, after all, been introduced; and to test this I employed a photograph on which wave-lengths 38 and 40 were taken as extreme points, and the distance between them divided into twenty portions in like manner. I then found that, although the point 39 was very slightly changed, all the rest of Cornu's measurements between 39–40 fitted the photograph very well. That is, I found that the ten points between 39, as thus determined, and

* Mr. Rutherfurd, who has carefully studied these phenomena, ascribes them to the so-called "Talbot Bands."

40 fell as near the places assigned by CORNU as could possibly be expected, considering the great difference in the scale employed, and the vast increase in the details obtained by the grating. The new place for 39, which lies at 38·992 on CORNU's map, was finally settled as the start-point. The distance from 39 to 40 on the glass plate was carefully divided into 1000 parts; a scale was made in ink, and from the glass plate, which now contained both the spectrum and the scale, an enlargement on paper was obtained slightly greater than the map to be produced from it.

I willingly here express my great obligation to M. CORNU'S work, and add a tribute of admiration to its value and completeness. The way in which it bears the severe test put on it by the larger scale I have employed is marvellous.

C. CONSTRUCTION OF THE MAP.

Being thus in possession of an enlarged photographic print on which are marked the absolute wave-lengths, the next thing done was to study it side by side with a refraction photograph brought up to the same scale. Owing to the cause I have before referred to, it was found that, though there was no difficulty in recognising the chief lines in both, still in the diffraction photograph the smaller details were in many places quite different, and in many others very difficult to harmonise, the intensities of the lines having been greatly changed. Although, therefore, I was able to use the diffraction photograph for the positions of the chief lines, I had to depend on the refraction photograph for the detailed work and the intensity of the lines.

A trial map on the scale adopted was then very carefully made by the aid of the photographs and the original standard refraction negative. An enlargement was then made on three times the scale, =twelve times the scale of ÅNGSTRÖM.

I tried several methods of conveniently comparing the drawing with the photograph, after the chief lines had been inserted in positions indicated on the scaled negative. The following I found most satisfactory, and, indeed, it has been of great service in the construction of the map.

The board on which the large-scale map was drawn was placed in an upright position, and at about 10 feet in front of it a small sighting aperture of 3 millims. diameter was adjusted.

The diffraction photograph with its accompanying scale was supported between the board and this aperture, care being taken to keep the line joining the eye-hole and the centre of the map at right angles to the plane of the map, and the plane of the photograph parallel to the same plane. The position of the photograph was so adjusted that the lines on the photograph and the map, thus viewed from the observing aperture, were absolutely continuous. In this manner I was able to correct any error in the positions of the lines in the enlarged map with the greatest certainty. The details and intensities were added afterwards by mounting the refraction photograph in the same manner. Having thus corrected, checked, and finished the large map in ink, it

was sent to the School of Military Engineering at Chatham, where, by the kind permission of the authorities, it was photographed down to the scale determined upon.

IV. IMPORTANCE OF THE PHOTOGRAPHIC METHOD.

The importance of applying photography to the violet and ultra-violet portions of the solar spectrum seems to have been fully appreciated by Ångström.*

The classical "Spectre Normal" is avowedly incomplete about this region, both with regard to the number of lines and their wave-lengths. Even in portions of the spectrum distinctly visible—such, for example, as from G to F—a good photograph shows a much greater number of lines than the corresponding region of the "Spectre Normal." Similarly with regard to metallic spectra: the most complete spectral maps hitherto published, those of Thalèn,† contain in a given region a much smaller number of lines than are mapped for the same region by means of photography, even when the spectra are purified to the greatest possible extent by the elimination of all known impurity lines. Numerical comparisons illustrating this superiority of the photographic method over eye observation have already been given in the preliminary note before referred to.

Since the publication of the "Spectre Normal," and of Ångström and Thalèn's map of the violet portion of the solar spectrum, a map of the ultra-violet region has been constructed by Mascart‡ by means of photography; but this map possesses the disadvantage of an arbitrary scale, and no metallic lines are introduced. A diffraction spectrum obtained by means of photography was published by Dr. Draper in 1872. This spectrum extends from beyond G to O, and although the best diffraction spectrum hitherto published, the wave-length scale is too indistinct and blurred for use in very accurate measurements. The allocation of the lines with those of metallic spectra is, moreover, not shown, although the author states that he has photographed some of the metallic spectra, both by means of a grating and of a train of quartz prisms. The excellent photograph of the solar spectrum taken by Mr. Rutherfurd, as might have been expected, contains a much greater number of lines than the "Spectre Normal" for the same region, but being a refraction spectrum is of course not available for the purpose of directly determining wave-lengths. The most perfect map of the violet and ultra-violet solar spectrum at present in existence is that recently published by Cornu, who has determined absolutely the wave-lengths of thirty-six of the principal lines in the portion of the spectrum included between O (wave-length 3440) and beyond *h* (wave-length 4120), the remaining lines, about 650 in number, being introduced by interpolation. This map was constructed by means

* "Quant à la lumière violette et extra-violette, j'espere pouvoir, à l'aide de la photographie, en publier bientôt un aperçu exact et détaillé, surtout comme les tentatives, déjà faites à cet égard par M. Thalèn, confirment ces espérances."—('Recherches sur le Spectre Solaire,' Upsal., 1869.)

† 'Nova Acta Regiæ Societatis Scientiarum.' Upsaliensis, 1868.

‡ "On the Rays of the Ultra-violet Solar Spectrum." 'Compt. Rendus,' Nov., 1868.

of photography, and although very incomplete so far as regards metallic coincidences, it has proved of invaluable service in the construction of the present map.

That the value of the photographic method has not been over-estimated will be rendered evident from the following considerations.

In the first place, it is well known that a great extent of the spectrum totally invisible under ordinary circumstances can be recorded by means of photography. Thus, taking the length of spectrum which can be conveniently mapped by eye to extend from *h* (wave-length 4100) to near A (wave-length 7310), this map would be, on ÅNGSTRÖM's scale, 3·20 metres long. Taking about Q (wave-length 3177, MASCART) as the most refrangible limit of the spectrum conveniently workable by photography, we thus gain from this point to *h*, on the same scale, an additional length of nearly 1 metre.

Next with regard to the amount of detail obtained from photographs compared with that observable by eye. To illustrate the advantage of the present method, it will be instructive to compare a given region mapped by eye and by photography. The portion first selected has been mapped by ÅNGSTRÖM, by CORNU, and likewise in the course of the present work.

Solar Spectrum, Section 4000–4100.

Eye observation.	No. of lines.	Photographic record.	No. of lines.
"Spectre Normal"	32	CORNU's map	104
		Map constructed from RUTHERFURD's print of the solar spectrum	297
		Map constructed from negatives taken July, 1875 (including lines glimpsed merely)	333

In a region more easily visible than the above, the same superiority is exhibited by the photographic spectrum.

Solar Spectrum, Section 4200–4300.

Eye observation.	No. of lines.	Photographic record.	No. of lines.
"Spectre Normal"	94	Map constructed from RUTHERFURD's print of the solar spectrum	275
		Map constructed from a negative taken July 29, 1875 .	412

Seeing the great advantages thus offered by photography, even in a part of the spectrum distinctly visible, I am persuaded that it will be necessary to re-map the

whole length of spectrum capable of being photographed, before the conditions set forth at the commencement of this paper are fulfilled. Experiments made in the course of the present work show that, with a bromo-iodized collodion, it is possible to photograph as far as E in the green (wave-length 5269), with exposure of from two to three minutes. Neither can there be much doubt that, in a short time, we shall be in possession of methods enabling us to photograph any part of the spectrum with equal facility.

The importance of the photographic method will be still further enforced, if we consider that the maps of metallic spectra at present in existence, although constructed with the greatest skill, are necessarily incomplete, and are, moreover, not free from impurity lines. Before we can hope to arrive at any great generalisation in the theory of the spectrum, it is obvious that we must have pure spectra to deal with; and, as I have previously shown, the photographic method is the only one which enables such a purification to be effected.

V. PHOTOGRAPHIC PROCESS EMPLOYED.

The silver bath employed contains 40 grains of silver nitrate to the ounce of distilled water. It is made as neutral as possible.

Amongst many developers which were tried, I have found that used by CORNU to be best for these investigations, viz.:—

1 litre	Distilled water.
60 cubic centimetres . .	Saturated solution of ferrous sulphate.
30 „ „ . .	Glacial acetic acid.
30 „ „ . .	Alcohol.

In course of these researches I have used many collodions, but BLANCHARD'S extra bromo-iodized is by far the best and most suitable for spectrum photography, as its power for imparting detail is greater than any other, and the granular appearance under magnification is less than that produced by others, although this defect is still great, and leaves much to be desired.

The emulsion process and also the dry plate processes have yet to be more thoroughly tested in connexion with this work, but at present the time of exposure required has proved a fatal objection.

Having obtained the negative, whether from the grating or a train of prisms, the next process is to produce the necessary enlargement. I have found no lens so well suited for these enlargements as Mr. DALLMEYER'S rapid rectilinear. The working positive employed is a little below the scale of the map. The exposure required for transparencies is not so long as for a negative by about one-third. The intensifying solution should be passed over once, merely to secure the fine lines. The cyanide solution should be made rather weak for this work.

It is most important that both the negative and transparency should not be varnished, as the texture of the varnish will, when photographed, mar all fine detail.

VI. CONCLUSION.

In conclusion, I would remark that the accompanying map can only be regarded as provisional, seeing that when greater dispersion is employed, or a grating without the glass plate in front of it is available, much finer details will no doubt be revealed, and wave-lengths of all lines will be directly given.

With regard to the detail shown, I believe that the map as nearly represents the actual state of the solar spectrum at the epoch as can be expected with the photographic materials and dispersion at my disposal. There are some few lines which, when magnified, seem to show indications of being double, in some cases by a line leaning off on one side, in others by a trace of a central division. Such lines are, however, represented in the map as single, and the words "probably double" introduced opposite to them in the tables.

I may also mention that in other cases a solar line, although single under the highest magnifying power, has also been suspected to be double, because in my work on metallic spectra the supposed components have been found to be coincident with two lines in the spectra of two distinct metals, which lines just graze each other without actually coinciding.

The coincidence of the solar lines with those due to the vapours of the metals in this region I shall deal with in another paper.

In the annexed plate I have not only given the map, as reduced from the photographs in the manner described, but introduced a permanent enlargement of one of the photographs. It has not been possible to represent the intensities absolutely, on account of the varying tones of the absorption-lines themselves. The untouched photograph will allow of the detection of any errors of this nature.

I must express my obligation to the authorities of the School of Military Engineering, Chatham, for the permission they granted to have the enlargements made, and the reduced copy of the map photographed, there.

It is my duty and pleasure to record my deep obligations to my assistants, Messrs. MELDOLA, ORD, and STARLING for the care, patience, and skill they have successively shown in carrying on the various branches of the work. My thanks are also due to Corporals MURRAY and EWINGS, R.E., for their aid in enlarging the sketch maps and comparing them with the original negative.

In the map the lines have been fined down as much as possible to show the double lines, and the shading near H and K reduced to a minimum to show the finer lines in those regions.

TABLE I.—Fundamental lines used in the construction of Curves.
Section 3900–4000.

λ CORNU.	λ ÅNGSTRÖM.	λ Curve A from Photograph A.	λ Curve B from ROWLAND'S print.	Error λ CORNU − λ A.	Error λ CORNU − λ B.	State in CORNU'S Map.	State in ÅNGSTRÖM'S Map.	State in Standard Photograph.	Intensity in Solar Spectrum 1 = darkest.	Remarks.
3897·0 (Fe.)	Absent.	3897·00	..	0	..	Single : winged	Absent	Single ..	1	
3898·4 (Fe.)	„	3898·63	..	−·23	..	„ „	„	„ ..	1	
3902·0 (Fe.)	„	3902·10	..	−·1	..	„ „	„	„ ..	1	
*3904·8	.. „	3904·77	..	+·03	..	„ „	„	„ ..	1	
3909·4	.. „	3909·23	..	+·17	..	Single ! . .	.. „	Double (!)	3, 2**	**Probably double. Identity with CORNU'S line doubtful.
3913·4	.. „	3913·82	..	−·42	..	„ ..	.. „	Double (?)	2, 2**	**Probably double.
3916·6	.. „	3916·50	..	+·1	..	„ ..	.. „	Single ..	2	
3918·4 (Fe.)	„	3918·72	..	−·32	..	„ ..	.. „	„ ..	2	
3920·0 (Fe.)	„	3920·00	..	0	..	„ ..	.. „	„ (!)	1	Probably double.
*3922·1 (Fe.)	„	3922·27	..	−·17	..	„ ..	.. „	„ ..	1	
3927·2 (Fe.)	„	3927·30	..	−·10	..	„ ..	.. „	„ ..	1	
3929·8 (Fe.)	„	3929·77	..	+·03	..	„ ..	.. „	„ ..	1	
3935·2	.. „	3935·17	..	+·03	..	„ ..	.. „	Double ..	3, 3	
3937·8	.. „	3937·72	..	+·08	..	„ ..	.. „	Single .	2	
3940·0	.. „	3940·43	..	−·43	..	„ ..	.. „	„ ..	2	
3941·8	.. „	3941·80	..	0	..	„ ..	.. „	„ ..	2	
3943·1	.. 3942·70	3943·30	..	−·20	..	„ winged	Single	„ ..	1	
3946·9	.. Absent	3947·00	..	−·10	..	„ ..	.. Absent	Triple ..	3, 3, 3	
3947·9	.. „	3947·85	..	+·05	..	„ ..	.. „	Single (!)	2	Probably triple.
3950·0	.. „	3950·22	..	−·22	..	„ ..	.. „	„ ..	2	
*3952·0	.. 3951·50	3951·72	..	+·28	..	„ ..	.. Single	Triple ..	2, 2, 2	
3955·0	.. Absent	3954·50	..	+·5	..	„ ..	.. Absent	Single ..	2	
3955·8 (Fe.)	„	3955·50	..	+·3	..	„ ..	.. „	Triple ..	2, 2, 2	
3959·2	.. „	3959·48	..	−·28	..	„ ..	.. „	Single ..	3	
3960·5	.. 3961·20	3960·50	..	0	..	„ winged	Single	„ ..	1	
3963·6	.. Absent	3963·60	..	0	..	„ ..	.. Absent	Double ..	3, 3	
3965·0	.. „	3965·05	..	−·05	..	„ ..	.. „	„ ..	3, 2	
3965·8	.. „	3965·80	..	0	..	„ ..	.. „	„ ..	2, 2	
3968·7	.. „	3968·40	..	+·3	..	„ ..	.. „	Single ..	1	
3970·4	.. „	3970·32	3970·45	+·08	−·05	„ ..	.. „	„ ..	2	
3972·0	.. „	3972·00	3972·80	0	+·2	„ ..	.. „	„ (!)	1	Probably double.
3975·5	.. „	3975·27	3975·33	+·23	+·17	„ ..	.. „	Double (!)	3, 3**	**Probably double.
3978·9	.. „	3978·74	3979·00	+·16	−·1	„ ..	.. „	„ ..	2, 4	
3981·0	.. „	3981·24	3981·20	−·24	−·2	„ ..	.. „	„ (!)	2, 3**	**Probably double.
3983·7	.. „	3983·15	3983·40	+·55	+·30	„ ..	.. „	Single ..	1	
*3986·1	.. 3986·0	3986·18	3986·36	−·08	−·26	Double ..	.. Single	Double ..	1, 1	
3989·0	.. Absent	3989·12	3989·16	−·12	−·16	Single ..	.. Absent	Single ..	1	
3993·25	.. „	3993·60	3993·30	−·35	−·05	Single ..	.. „	Double ..	2, 2	
3997·5	.. 3997·8	3997·72	3997·33	−·22	+·17	„ ..	.. Single	„ ..	1, 2	
3999·8	.. Absent	3999·80	3999·30	0		„ ..	.. Absent	„ ..	2, 2	A line in ÅNGSTRÖM'S Map at wave-length 4001·4 not capable of being identified.
4000·7	.. „	..	4001·24	..	−·54	„ ..	.. „	Single ..	2	
4003·3	.. „	..	4003·40	..	−·1	„ ..	.. „	„ ..	3	
*4004·4 (Fe.)	4004·7	..	4004·72	..	−·32	„ ..	.. Single	Double ..	1, 1	
4006·0	.. Absent	..	4006·28	..	−·28	Double ..	.. Absent	„ ..	2, 2	
4007·4	.. „	..	4007·56	..	−·16	Single ..	.. „	Single ..	4	
4008·6	.. „	..	4008·50	..	+·1	„ ..	.. „	„ ..	2	
4009·40	.. „	..	4009·25	..	+·15	„ ..	.. „	Double ..	4, 1	

* Lines marked thus are those measured absolutely by CORNU. Lines marked (Fe.) are iron lines, assigned to this metal by CORNU, which have been identified by a comparison photograph of Fe and solar spectrum.

N.B.—In the case of lines which the standard photograph resolves into groups of two or more, the centre is always taken for measurement. In the case of multiple lines, the intensities are placed in the same order as the lines are seen when looking at the map, i.e., the more refrangible on the observer's left. The scale of intensities adopted is the same as that employed by THALÉN for the metallic spectra—1 being the darkest and 5 the lightest. In the overlapping portions of curves A and B the values given by B are adopted.

Table II.—Lines interpolated from Curves.
Section 3900–4000.

λ Cornu.	λ Ångström.	λ Curve A.	λ Curve B from Rutherfurd's print.	Error λ Cornu − λ Curve.	State in Cornu's Map.	State in Ångström's Map.	State in Standard Photograph.	Intensity in Solar Spectrum 1 = darkest.	Remarks.
Absent ..	Absent	3900·00	..	..	Absent	Absent	Double ..	3, 3	
3900·6 ..	,,	3900·60	..	0	Single	,,	Single ..	2	
Absent (1)	,,	3901·25	..	..	Absent ?	,,	,, ..	3	No distinct line in Cornu's map.
3903·0 ..	,,	3903·16	..	−·16	Single ..	,,	,, ..	2	
3904·3 ..	,,	3904·08	..	+·22	,, ..	,,	,, ..	3	
3905·9 ..	,,	3905·80	..	+·16	,, ..	,,	,, ..	1	
3907·1 (1) ..	,,	3907·17	..	−·07	,, ..	,,	,, ..	4	Identity with Cornu's line doubtful.
3907·9 ..	,,	3908·10	..	−·20	,, ..	,,	Double ..	3, 4	
3909·9 (1) ..	,,	3909·87	..	+·03	Double ?	,,	,, ..	3, 3	
3911·6 ..	,,	3911·30	..	+·30	Single ..	,,	Triple ..	3, 3, 3	
3912·9 ..	,,	3912·80	..	+·10	,, ..	,,	Double ..	2, 2	
3915·2 ..	,,	3915·07	..	+·13	,, ..	,,	Quadruple	4, 3, 3, 3	
3917·8 ..	,,	3917·85	..	−·05	,, ..	,,	Single ? ..	2	Probably double. Represented in Cornu as beginning a broad band.
3921·4 ..	,,	3921·28	..	+·17	,, ..	,,	Triple ..	3, 3, 3	Cornu's line in midst of shading.
3924·0 ..	,,	3923·85	..	+·15	,, ..	,,	Single ..	3	
3924·3 ..	,,	3924·42	..	−·12	,, ..	,,	,, ..	3	
3925·3 ..	,,	3925·28	..	+·02	,, ..	,,	,, ..	2	
3928·5 ..	,,	3928·62	..	−·12	,, ..	,,	Double ? ..	3, 4**	**Probably double.
Absent ..	,,	3931·10	..	..	Absent	,,	Single ..	3	
Absent ..	,,	3936·72	..	..	,, ..	,,	,, ..	4	
Absent ..	,,	3939·50	..	..	,, ..	,,	,, ..	5	
Absent ..	,,	3944·61	..	..	,, ..	,,	,, ..	3	Centre of a group of four, represented in Cornu by a single line at 3944·2.
Absent ..	,,	3945·15	..	..	,, ..	,,	,, ..	5	
3946·2 (1) ..	,,	3946·28	..	−·08	Single ..	,,	,, ..	3	Identity with Cornu's line doubtful.
3948·9 ..	,,	3949·10	..	−·20	,, ..	,,	,, ..	3	
Absent ..	,,	3952·65	..	..	Absent	,,	,, ..	4	
Absent ..	,,	3957·60	..	..	,, ..	,,	,, ..	4	
3962·1 ..	,,	3962·20	..	−·10	Single ..	,,	,, ..	2	
3971·1 (1) ..	,,	..	3971·40	−·30	,, ..	,,	,, ..	3	Identity with Cornu's line doubtful.
3974·0 (1) ..	,,	..	3974·13	−·13	,, ..	,,	,, 1..	2	Probably double. Identity with Cornu's line doubtful.
3976·9 (1) ..	,,	..	3977·18	−·28	,, ..	,,	,, ..	2	Identity with Cornu's line doubtful.
3984·7 ..	,,	..	3984·80	−·10	,, ..	,,	,, 1..	2	Probably double.
3988·5 ..	,,	..	3988·50	0	,, ..	,,	,, ..	2	
3990·3 ..	,,	..	3990·49	−·19	,, ..	,,	,, ..	3	
3991·8 (1) ..	,,	..	3992·25	−·45	,, ..	,,	,, ..	3	Identity with Cornu's line doubtful.
3995·0 (1) ..	,,	..	3994·63	+·37	,, ..	,,	,, ..	1	Identity with Cornu's line doubtful.
3995·80 ..	,,	..	3995·25	+·55	,, ..	,,	,, ..	3	Identity with Cornu's line doubtful.
3996·7 ..	,,	..	3996·75	−·05	,, ..	,,	,, ..	1	
Absent ..	,,	..	3998·35	..	Absent	,,	,, ..	3	

N.B.—In the case of multiple lines the same arrangement holds good as in the last table.

TABLE III.—Table of Wave Lengths 390,000 to 400,000 thousand millionths of millim.

Wave Length.	Intensity in Sun 1 = greatest.	Wave Length.	Intensity in Sun 1 = greatest.	Wave Length.	Intensity in Sun 1 = greatest.
390,000	1	391,518	3	393,435	5
390,023	4	391,555	3	393,475	3
390,048	3	391,570	3	393,495	3
390,105	3	391,605	3	393,628	3
390,135	2	391,648	2	393,672	4
390,162	4	391,675	5	393,718	3
390,180	4	391,700	5	393,745	2
390,220	4	391,725	5	393,885	5
390,245	1	391,754	2	393,920	5
390,270	3	391,773	2	393,948	5
390,290	5	391,815	5	393,975	2
390,335	2	391,835	2	394,026	3
390,373	5	391,865	5	394,050	3
390,423	3	391,882	5	394,080	3
390,441	4	391,915	5	394,136	5
390,478	4	391,945	1	394,149	2
390,500	1	391,978	3	394,178	5
390,535	4	391,991	3	394,223	3
390,585	2	392,008	2	394,250	3
390,600	1	392,025	3	394,273	3
390,622	3	392,050	3	394,300	1
390,650	5	392,080	3	394,368	3
390,673	5	392,123	5	394,381	3
390,692	4	392,165	5	394,422	3
390,712	5	392,180	5	394,440	3
390,732	4	392,200	1	394,481	4
390,765	5	392,250	5	394,515	5
390,785	5	392,265	5	394,568	5
390,810	3	392,295	5	394,600	3
390,825	4	392,338	4	394,650	3
390,875	5	392,368	3	394,665	3
390,900	3	392,438	3	394,680	3
390,922	2	392,488	3	394,720	3
390,968	3	392,523	2	394,731	5
390,985	3	392,555	4	394,776	2
391,010	3	392,568	4	394,815	3
391,025	3	392,615	5	394,885	3
391,040	5	392,665	4	394,938	3
391,075	5	392,700	1	395,012	2
391,102	3	392,735	3	395,038	5
391,115	3	392,755	3	395,055	5
391,128	3	392,762	3	395,070	5
391,158	3	392,785	3	395,092	4
391,223	3	392,830	3	395,110	5
391,258	4	392,845	4	395,130	5
391,275	2	392,883	4	395,160	2
391,290	2	392,900	3	395,190	2
391,335	4	392,950	1	395,210	2
391,360	2	393,000	4	395,248	4
391,375	2	393,023	3	395,278	4
391,400	5	393,100	3	395,295	5
391,425	5	393,165	3	395,338	4
391,451	4	393,221	4	395,365	4
391,480	4	393,270	1	395,385	5
391,492	3	393,325	3	395,423	2

TABLE III. continued.—Table of Wave Lengths.

Wave Length.	Intensity in Sun 1 = greatest.	Wave Length.	Intensity in Sun 1 = greatest.	Wave Length.	Intensity in Sun 1 = greatest.
395,450	5	397,148	4	398,598	1
395,472	5	397,155	4	398,635	1
395,498	3	397,200	4	398,678	5
395,530	3	397,238	4	398,695	3
395,548	3	397,275	1	398,728	5
395,568	3	397,305	5	398,742	5
395,600	2	397,338	3	398,768	5
395,668	3	397,361	2	398,800	4
395,700	3	397,382	2	398,848	2
395,722	2	397,480	3	398,922	1
395,748	5	397,448	4	398,972	2
395,775	4	397,478	4	398,985	5
395,820	5	397,495	4	399,045	3
395,850	5	397,535	3	399,065	4
395,875	5	397,555	1	399,100	3
395,920	3	397,578	3	399,155	3
395,935	5	397,625	5	399,172	5
396,028	3	397,685	2	399,215	2
396,055	1	397,722	5	399,238	3
396,115	3	397,741	3	399,250	5
396,130	3	397,760	3	399,274	5
396,172	4	397,785	3	399,288	5
396,182	3	397,810	5	399,325	2
396,215	2	397,830	5	399,348	2
396,242	5	397,868	2	399,375	2
396,268	3	397,880	4	399,398	3
396,330	3	397,915	4	399,465	1
396,362	3	397,975	4	399,498	5
396,450	3	398,000	4	399,520	3
396,508	2	398,023	4	399,533	4
396,525	2	398,038	4	399,570	3
396,550	2	398,065	5	399,610	5
396,571	2	398,083	2	399,625	5
396,662	4	398,105	3	399,648	2
396,741	1	398,150	3	399,665	3
396,832	2	398,168	3	399,688	1
396,865	5	398,215	3	399,728	1
396,882	5	398,240	3	399,750	2
396,911	4	398,273	5	399,778	5
396,925	3	398,320	1	399,798	2
396,948	3	398,345	3	399,848	3
396,976	5	398,360	3	399,875	5
397,000	5	398,385	3	399,920	5
397,035	2	398,423	3	399,958	2
397,078	5	398,465	2	399,977	2
397,120	3	398,550	2		

XIII. *Researches on the Minute Structure of the Thyroid Gland.*

By E. Cresswell Baber, *M.B. Lond.*

Communicated by E. Klein, *M.D., F.R.S.*

Received November 25, 1880,—Read January 6, 1881.

[Plates 68-69.]

Contents.

	Page
Introduction	577
Bibliography	578
Macroscopic appearances	578
Methods employed	580
Microscopic appearances	582
I. Vesicles	582
II. Epithelium, reticulum, and basement membrane	585
III. Contents of vesicles	589
IV. Parenchyma	596
V. Lymphatics and their contents	597
VI. Blood-vessels	600
VII. Undeveloped portions	600
General conclusions	602
Explanation of the plates	604

INTRODUCTION.

The following research is a continuation of that previously communicated to the Society on the "Minute Anatomy of the Thyroid Gland of the Dog," and published in the Philosophical Transactions for 1876 (Vol. 166, Part II.). It is based on a histological examination of the thyroid gland in the following vertebrate animals:—

I. *Mammalia.* Dog, Cat, Horse, Rabbit, Ox, Sheep, Seal.

II. *Aves.* Pigeon, Fowl, Rook.

III. *Reptilia.* Tortoise.

IV. *Amphibia.* Frog.

V. *Pisces.* Skate, Conger Eel.

(Altogether the glands of more than sixty animals have been examined in this research.)

My examination of the human thyroid gland is unfortunately not sufficiently com-

plete to allow of its being included in this communication, partly owing to its great delicacy of structure, and partly to the difficulty of obtaining this gland perfectly fresh, a condition which is absolutely necessary in order to examine its minute anatomy. I shall therefore merely refer to this gland incidentally on one or two points in the course of the paper.

BIBLIOGRAPHY.

The following are the chief publications to which it will be necessary to refer in the present communication :—

LEYDIG : 'Lehrbuch der Histologie.' 1857.

HENLE : 'Handbuch der Systematischen Anatomie des Menschen,' vol. ii. 1866.

KÖLLIKER : 'Handbuch der Gewebelehre.' 5th edition, 1867.

PEREMESCHKO : "Ein Beitrag zum Bau der Schilddrüse." Zeitschrift für Wissenschaftliche Zoologie, xvii. 1867.

ROLLESTON : 'Forms of Animal Life.' 1870.

VIRCHOW : 'Pathologie des Tumeurs.' French translation. Lect. 22, vol. iii. 1871.

W. MÜLLER : Jenaische Zeitschrift für Medizin und Naturwissenschaft, vi. 1871.

VERSON : Article "Thyroid Gland," in STRICKER'S 'Human and Comparative Histology.' Sydenham Society's translation, by H. POWER, vol. i. 1871.

P. A. BOÉCHAT : 'Recherches sur la Structure Normale du Corps Thyroïde.' Thèse. Paris, 1873.

FREY : 'Histology and Histo-Chemistry.' 4th edition, translated by BARKER. 1874.

BABER : "Contributions to the Minute Anatomy of the Thyroid Gland of the Dog." Philosophical Transactions of the Royal Society, Vol. 166, Part II. 1876.

O. ZEISS : 'Mikroskopische Untersuchungen über den Bau der Schilddrüse.' Inaugural Dissertation. Strassburg, 1877.

HUXLEY and MARTIN : 'Practical Biology.' 4th edition, 1877.

KLEIN : "Observations on the Structure of Cells and Nuclei." Quarterly Journal of Microscopical Science, vol. xviii., new series. 1878.

MACROSCOPIC APPEARANCES.

Before entering on the histology of the gland, it may be well to mention the following points in regard to its macroscopic anatomy which have been noted in this research (compare SIMON, "On the Comparative Anatomy of the Thyroid Gland," Phil. Trans., 1844; and OWEN, 'Anatomy and Physiology of the Vertebrata,' vol. i., 1866) :—

Mammalia.—In some of these (such as the Dog, Sheep, &c.) the gland consists of two distinct halves, situate one on either side of the windpipe. In other instances the two lateral halves are united across the median line. For example, on injecting one lobe of the thyroid of a Rabbit (11 weeks old) by the method of puncture, what

appeared to be a broad band of small lymphatics became injected, running across the trachea to the lower part of the opposite lobe. Sections of this connecting band, examined microscopically, showed that it contained a considerable quantity of gland-tissue, and was therefore a true isthmus. In the Kitten, on injecting one gland by the puncture method with Berlin blue solution, what was apparently a large lymphatic vessel became filled, running across the trachea to the lower part of the opposite gland. This probably also formed a delicate isthmus of gland-tissue, connecting the two lobes.

Aves.—In the Birds examined (Fowl, Rook, Pigeon) the thyroid gland was composed of two small round or elongated bodies, situate in the upper part of the thorax, and in close connection with the jugular vein and carotid artery on either side.

Reptilia.—In the Tortoise the gland is single, and forms a yellowish, rounded, somewhat flattened organ, situate just above the base of the heart, between the right and left aorta.

Amphibia.—*Frog.* The following extracts will show that observers are by no means unanimous in regard to the position and character of the thyroid gland in this animal.

Leydig (*op. cit.*, p. 376) says that the thyroid gland of tail-less Batrachians (Frogs and Toads) usually consists of only three large vesicles, provided with a fine capillary network and isolated from one another, whose contents are neither clear fluid nor colloid, but a finely granular and partly fatty substance. Rolleston (*op. cit.*, pp. 184, 185) describes and figures the thyroid glands as placed just internally to the jugular veins. Huxley and Martin (*op. cit.*, p. 181), on the contrary, state that "the thyroid gland appears to be represented by two or more oval bodies, which are found attached to the lingual vessels, and between the aortic and pulmo-cutaneous trunks." W. Müller (*op. cit.*, p. 438) found that in young Frogs the thyroid lay on the two sides of the body of the hyoid bone, close in front of the point of attachment of the *cornua thyreoidea*. It was surrounded on all sides by muscles, and possessed a thin connective-tissue capsule, from which the vessels with their connective-tissue adventitia stretched into the interior. The gland substance consisted entirely of closed vesicles, with a delicate membrana propria and a single layer of cubical epithelium without pigment, and a central cavity containing a transparent colourless fluid, without any structural elements.

For some time I was unable to find in the Frog any body corresponding histologically to the thyroid gland, but subsequently, by following W. Müller's directions as just quoted, I discovered it in transverse sections of the head. My observations show that in the Frog there are two bodies presenting the structure of the thyroid gland. They are situate, one on either side, on the ventral surface of the hyoid cartilage (or bone), being usually, but not always, in direct contact with this structure. They are found either between the hyoid cartilage and the hyoglossus muscles, or else immediately to the outer side of these muscles, on a level just anterior to the

point of their convergence. In quite small Frogs the thyroid gland appears in transverse sections as a somewhat flattened body, lying on the ventral surface of the hyoid cartilage, partly between it and the hyoglossus muscle. In rather larger Frogs it presents in transverse section a somewhat triangular outline (see Plate 68, fig. 1, 1, 1), the base being applied to the ventral surface of the cartilage (Plate 68, fig. 1, 1) and the apex projecting forwards by the side of the hyoglossus muscle. In still larger Frogs transverse sections show that the gland often extends forwards, but its development on the two sides appears to be irregular and unsymmetrical. In Frogs of full size it may, however, still be found flattened and extending very slightly forwards.

I may here mention that the structure of this gland resembles that of the thyroids of other animals. It consists of numerous vesicles separated by a stroma of connective tissue, the whole being surrounded by a capsule of similar character. The vesicles are lined by a single layer of cubical or slightly columnar epithelial cells, which are devoid of dark pigment. It will be seen that the results of these observations on the Frog agree on the whole with those of W. MÜLLER.

Pisces.—In the Skate, as far as I have seen, the gland is single (with the exception of a few detached vesicles) and forms a yellow flattened, lobulated body, occupying the median line at the bifurcation of the branchial artery.* Anteriorly it sometimes presents a narrow process of gland-tissue running forwards, and behind it is limited by the bifurcation of the branchial artery. The vesicles of the gland, of various sizes, are distinctly visible to the naked eye. In the Conger Eel the gland occupies a similar position and forms a reddish flattened body.

METHODS EMPLOYED.

Methods of Injection.

I. The gland of a freshly-killed animal was injected *in situ* with a solution of Berlin blue by the method of puncture. The gland being hardened first in a mixture of equal parts of methylated spirit and water, and afterwards in methylated spirit, and sections mounted in dammar or balsam in the usual way.

II. The fresh gland was injected in a similar manner with $\frac{1}{8}$ per cent. solution of nitrate of silver in water and hardened as before.

III. The fresh gland was injected *in situ* by the method of puncture with $\frac{1}{2}$ per cent. solution of osmic acid in water. The gland being removed was cut up into small pieces, and these placed in solutions of the same acid ranging in strength from $\frac{1}{10}$ to 1 per cent. for periods ranging from 1 to 24 hours. The gland being subsequently placed in alcohol and sections mounted in glycerine or in dammar.

* HANDFIELD JONES (quoted by OWEN, *loc. cit.*) has observed in the Skate a second body of similar structure situated some distance behind this. I have not observed this second gland, but cannot deny its existence.

IV. In the Tortoise a double injection of the gland was made by filling the blood-vessels from the aorta with GERLACH'S carmine mass, and after about 20 minutes injecting the lymphatics of the gland *in situ* by the method of puncture with Berlin-blue solution—the latter, however, did not run very well.

In several animals also the blood-vessels only were injected with a solution of Berlin blue.

Methods of Hardening.

The method employed in by far the larger number of cases was that of placing portions of the fresh gland (injected or uninjected) into a mixture of equal parts of methylated spirit and water for a period varying from a few days to a couple of weeks (or longer), and afterwards into pure methylated spirit until sufficiently hard for cutting, the hardening being sometimes completed by immersion in absolute alcohol, although this is not necessary. Sections of the hardened gland were then stained with hæmatoxylin or picrocarminate of ammonia and mounted in dammar or balsam. The following methods were serviceable for showing special points in the structure of the gland (which will be referred to subsequently).

I. Thyroid of Sheep or Tortoise hardened in a $\frac{1}{2}$ to 1 per cent. solution of chromic acid for 17 to 25 hours and subsequently in alcohol, for showing the intercellular reticulum.

II. Thyroid of Tortoise hardened in MÜLLER'S fluid for four days, stained with hæmatoxylin and mounted in glycerine, for showing the layer of connective tissue on the epithelial wall.

The following methods gave specimens which were either inferior, showing nothing fresh, or else were entirely useless:—(*a*) Immersion in a saturated solution of picric acid for 18 hours, afterwards in a strong solution of gum for 18 hours, and, finally, in methylated spirit. This method was only tried in the Ox, in which animal, it is only fair to state, the results obtained by other methods were very disappointing. (*b*) Solution of monochromate of ammonia ($2\frac{1}{2}$ and 5 per cent.), subsequent hardening in spirit. (*c*) Immersion for two days in a mixture of two parts of chromic-acid solution ($\frac{1}{6}$ per cent.), and one part of methylated spirit, afterwards in alcohol. The three methods recommended by KLEIN (*loc. cit.*) for showing intra-nuclear and intra-cellular reticulum were tried with the results mentioned below.

The general histology of the gland is certainly best shown in specimens hardened *in alcohol only*, although the facility with which the minute structure of this organ can be examined varies greatly in different animals, and also probably at different times in the same animal (*vide infra*, p. 586). BOÉCHAT (*op. cit.*, p. 13) also found hardening in alcohol the most successful method. The thyroid gland of no animal, as far as my experience goes, surpasses that of the Dog in the facility it presents for microscopic examination.

MICROSCOPIC APPEARANCES.

Speaking generally, the only points in regard to the histology of the thyroid gland, on which recent authors appear to be unanimous, are, that it consists of "cavities" lined by a single layer of epithelium and held together by a more or less dense stroma of connective tissue. Further, that numerous blood-vessels and lymphatics, also nerves, are found in the stroma, and that the whole organ is surrounded by a capsule of connective tissue, which is continuous with the stroma in its interior. On other points, such as the shape of these "cavities" (vesicles), the character of their epithelium, the nature of the contents of the vesicles, &c., the opinions of authors are at variance. In considering these subjects in order, it will be convenient to discuss them under the following heads.

Literature.—The majority of observers (*e.g.*, HENLE, KÖLLIKER, VERSON, FREY, and LEYDIG) have described the vesicles of the thyroid gland as consisting of closed globular bodies, not communicating with one another. VIRCHOW, and more recently BOÉCHAT and ZEISS, however, believe that this is not the case, but that the supposed closed globular bodies form a system of branched cavities in the gland. VIRCHOW (*op. cit.*, p. 201) finds that the apparently vesicular bodies are in multiple connection with one another. BOÉCHAT (*op. cit.*, p. 43) considers that all the cavities of the thyroid gland communicate with one another. ZEISS (*op. cit.*, p. 14) does not go to this length, but maintains that in the thyroid gland of young Children, of the Calf, of young Dogs and Sheep, a not insignificant part of the parenchyma is made up of branched cavities (branched tubes); but whether these all communicate with one another, and whether completely closed vesicles also occur, he leaves undecided. Both BOÉCHAT and ZEISS endeavoured to prove the existence of these branched cavities by injections by the method of puncture, but without success. Their failure they attribute to the injecting fluid entering the large spaces of the lymphatics or blood-vessels in preference to the branched cavities, which, as ZEISS points out, are filled partly with an easily coagulable, albuminous fluid and partly with the viscid colloid material. Although this observer has traced the epithelial layer running from one vesicle to another, he has never observed tubes passing from one lobule of the gland to another. ZEISS demonstrated these tubes by floating them out in a solution of common salt from small portions of the fresh gland of young animals, also after maceration in iodised serum or in dilute MÜLLER'S fluid. Circular, oval, conical, or pyramidal cavities were thus obtained together with long tubes, sometimes extending "right across the field of the microscope, having a narrow, wide, or irregular lumen, which divide and branch, and are provided at intervals with lateral protuberances or constrictions." He considers that the tubes, so plainly seen by isolation or maceration, are rendered invisible in sections by the circumstance that they interlace so freely in all directions that in a section they must

always be cut across. W. Müller (*loc. cit.*), who appears to agree with the majority of observers in considering that in the fully developed state the vesicles are closed globular bodies, describes an earlier stage (in the development of the gland) in which, before the formation of vesicles the gland consists of a network of cylindrical tubes, at first irregular, but afterwards becoming more regular in character. They are composed of a very fine envelope of connective tissue, and are filled in the interior with short cylindrical epithelial cells, placed radially on the envelope. The tubes are separated by processes of the mesoblast, which have grown inwards from the connective tissue surrounding the gland. The tubes are at first *solid*, but are subsequently provided with a central lumen. After the lumen in the interior of the tubes has become more marked, the separation of different segments of the tubes takes place by the growth into them laterally of processes of the mesoblast, which results in the formation of gland-vesicles. The same observer finds that in the human thyroid the shape of the vesicles varies much from the fifth month of embryonic life to the age of three years. From this period to puberty the changes occurring are, he considers, that the number of glandular deposits continues to diminish and the follicles increase in size, by the accumulation of secretion, which shows a great tendency to become thickened, and they approach the spherical shape. Peremeschko (*op. cit.*, p. 283) found the vesicles in young embryos regularly round; in mature embryos mostly of irregularly polygonal form; in young animals very seldom of regular round form, and still more rarely so in adult animals. In the latter the vesicles have, he says, the most different forms; they are almost always polygonally round or polygonally oval.

Observations.

My own observations on this subject may be summarised as follows:—

Dog.—In the gland of young Dogs (aged five weeks and three months respectively*) I have observed that the gland-vesicles are very much branched and present numerous hollow ramifications (see Plate 68, figs. 2 and 3). In order to see these well in sections of hardened glands it is necessary that the sections should be *thick*, and then by careful focussing, the vesicles with their usual contents can be traced ramifying both upwards and downwards, and laterally in all directions. In the glands of numerous other Dogs of different ages I have ascertained with equal certainty that the vesicles present very few, if any, ramifications. When the vesicles present these branches, the intervals between them form the inflexions projecting into the interior of the vesicles, which have been already described by Verson (*op. cit.*, p. 371) and myself (*op. cit.*, p. 560). I have examined fresh portions of the glands of young Sheep in salt solution and in Müller's fluid (according to the method recommended by Zeiss for floating out the tubes), but have obtained no appearances that might not be produced by the much-branched vesicles just described in the thyroid glands of young Dogs. In fact, there can be no doubt that these are the same structures as those which Zeiss

* The ages of most of the Dogs mentioned in this paper were only approximately ascertained.

obtained by flotation. Our observations also agree, inasmuch as ZEISS observed these tubes in the glands of *young* animals: at least he makes no statement to the effect that they have been found in the glands of adult animals. From the fact of their complete, or almost entire, absence in the glands of adult Dogs, I regard these much-branched vesicles merely as forming a stage in the growth of the organ, and consider that in its fully developed state the gland-vesicles form cavities more or less spherical in shape, which are very rarely, if at all, branched, or, in other words, in communication with one another. The presence of branched vesicles in a gland probably indicates that the number of vesicles is being increased by the growth into the vesicles already present of involutions of their walls, together with processes of the inter-vesicular tissue. PEREMESCHKO (*loc. cit.*) figures this division of the gland vesicles both in the embryo and adult Rabbit. He considers it probable that the division of vesicles takes place after intra-uterine life. W. MÜLLER is also of opinion, from an examination of the gland at different ages in Man, that completely-formed follicles (vesicles) undergo segmentation by the growth into them of processes of the mesoblast. I shall have occasion to refer to W. MÜLLER'S observations on the early stage in the development of the gland in speaking of the "undeveloped portions."

The above conclusions refer to the thyroid gland of the Dog, as it is in this animal that I have chiefly studied the shape of the vesicles.

In the thyroid gland of Birds (Rook, Fowl, Pigeon), as far as I have seen, the vesicles are, as a rule, small and very rarely branched. In the last-named bird (Pigeon) the larger vesicles are usually found near the surface of the gland.

In the Tortoise the vesicles are sometimes, but not usually, branched.

In the thyroid gland of full-sized Frogs the vesicles are usually large. In smaller Frogs their size is generally less. They are occasionally branched.

The gland-vesicles of the Conger Eel are often of large size and more or less branched. In the Skate the vesicles are also often large, and branched ones are sometimes seen.

In all these animals the exact age was unascertained.

Whilst, therefore, branched vesicles may occur in the thyroid gland of adult animals, they do not, as far as I have seen, communicate with one another to any extent—a conclusion which, it appears to me, is strongly corroborated by the fact that in the course of a large number of interstitial injections in the glands of different animals I have never succeeded in injecting any system of branched tubes. I cannot think with ZEISS that the viscid contents of the vesicles would have the effect of preventing the injecting fluid from entering them, when we know that the viscid contents of the lymphatics has no such effect.

The exact shape of the vesicles is of course of minor importance. My chief object in drawing attention to this point has been to show that in the adult state the vesicles form *closed cavities*, and do not consist of a system of tubes traversing the substance of the gland as BOÉCHAT suggests.

In this connection it may be well to mention the following appearance which I have observed in the thyroid gland of several Conger Eels:—In the walls of the vesicles, which in this animal are often of considerable size in proportion to the whole organ, small vesicles are seen, as shown in Plate 68, fig. 4. These secondary vesicles (Plate 68, fig. 4, s. s) are provided with an epithelial wall, and with contents similar to the primary vesicles. From the fact that by their growth they have evidently flattened out to a greater or less degree the epithelium separating them from the cavity of the primary vesicle, it is probable that the two layers of epithelium separating the cavity of the small, from that of the large vesicle sometimes give way, allowing the contents of the two vesicles to mix. At the same time others of these small vesicles probably grow outwards becoming independent structures (Plate 68, fig. 4, s. s). It is possible that some of these small vesicles may be curved branches of a large one which are cut across in the section.

II. *Epithelium, Reticulum, and Basement Membrane.*

The wall of the vesicle is composed of a layer of epithelial cells, and also, according to some observers, of a basement membrane and reticulum.

Results of previous observations.

Epithelium.—Recent observers are not yet in accord in regard to the shape of the epithelial cells. Verson (*loc. cit.*) describes the epithelial cells as higher than broad. Boéchat, on the contrary (*op. cit.*, p. 20), considers that in the normal state their breadth exceeds their height. Peremeschko (*op. cit.*, p. 281) is of opinion that in all animals, without exception, they are cylindrical, "but where the vesicles are enlarged, as in old animals, the epithelial cells present the most *bizarre* forms, which without doubt results from the mechanical pressure." Zeiss (*op. cit.*, p. 20) finds their shape in the Sheep and Calf highly cylindrical; in Man and in the Rat, shortly cylindrical; in the Cat, Dog, and Rabbit, cubical. He insists that the normal shape of these cells is not flattened but cylindrical. He also states that he has never found cubical and cylindrical cells in immediate proximity, but has not uncommonly seen vesicles with cylindrical cells next to those containing a cubical epithelium. Processes attached to the base of the epithelial cell have been observed by Peremeschko and Zeiss. Peremeschko noticed that the surface of the cell turned towards the cavity of the vesicle (summit of the cell) has a bright margin. Zeiss confirms this observation (on fresh cells examined in saline solution), and describes the bright margin as a cuticula.

Reticulum.—Zeiss (*op. cit.*, p. 23) has observed between the epithelial cells a fine reticulum, similar to that described by Von Ebner and Schwalbe in other glands provided with a cylindrical epithelium. He describes it as a network of delicate rods on the two surfaces of the epithelium with fine processes, or lamellæ, extending between the epithelial cells. In osmic and chromic acid preparations it presents

branched stellate figures, occasionally provided with nuclei having an indistinct contour. The reticulum extends on both aspects slightly beyond the level of the epithelium. He also describes in a profile view narrow rod-like structures between the cubical or columnar epithelial cells, resembling the club-shaped cells ("Keulenzellen") described by SCHWALBE in BRUNNER'S glands of the Dog.

Basement membrane.—KÖLLIKER, HENLE, and VERSON have described a very fine homogeneous membrane (*membrana propria*) lying outside the epithelial cells. W. MÜLLER also found a thin *membrana propria* in the vesicles of Man, Fowl, and Pig. FREY has been unable to see this membrane. PEREMESCHKO (*op. cit.*, p. 281), although he denies having seen a *membrana propria*, concludes that the cavities of the vesicles are formed by the epithelial cells, which are situated directly on the surrounding connective tissue, which forms a homogeneous, membranous limiting layer ("Grenzschicht"). BOECHAT also denies its existence, and considers that the epithelial cells are in direct contact with the endothelium of the lymphatics at a great number of points. The two walls (the endothelium and epithelium) are, he considers, separated at certain points by the bands of connective tissue which form the framework of the gland, and by capillary blood-vessels which ramify between them (p. 39). ZEISS (*op. cit.*, p. 15) has not been able to find any *membrana propria* of the above description, nor any structure resembling the "Drüsenkörbe" of BOLL. He has always observed between the endothelium of the lymphatics and the epithelium of the vesicles a very fine layer of connective tissue, in which he thinks that the capillaries ramify (p. 44).

Observations.

Epithelium.—The discrepancy of opinion of different observers in regard to the shape of the epithelial cells is, I think, attributable to three chief causes :—

1. The varying shape of these cells in different animals.
2. The different re-agents employed in examining and hardening the organ.
3. The state of functional activity of the gland at the time of removal.

With regard to re-agents I have chiefly examined these cells in specimens hardened in the usual way in alcohol—others, however, have been examined after being hardened with other re-agents, or in the fresh state. When it is found, however, that in the glands of the same species of animal prepared by *a similar method*, the appearance of the epithelial cells differs considerably, one is forced, I think, to admit the third factor above mentioned. Making due allowance for the two last-named elements of uncertainty, I have found that the general character of these cells in different animals is as follows :—

In the Dog, Cat, Kitten, Rabbit, Sheep, Rook, Pigeon, and Frog the epithelial cells are cubical or slightly columnar. In the Seal and Tortoise they are distinctly columnar. In the Horse, Skate, and Conger Eel they are highly columnar in shape.

The general tendency, therefore, of these cells throughout the vertebrate series, as

far as examined, appears to be to the columnar form—their exact height varying in different animals.

A double line of demarcation is often seen *on the summits* of the epithelial cells (*e.g.*, in the Tortoise). This doubtless corresponds with the cuticula described by Zeiss (*vide supra*, p. 585) and with the appearance of a membrane lining the *summits* of the epithelial cells which I described in the Dog (*op. cit.*, p. 560).

On examining with a moderately high power (Vérick, obj. 8) a transverse section of the wall of the vesicles in several animals, the appearance of a fine parallel striation has been observed in the epithelial cells running in the long axis of the cell, and extending for a greater or less distance from its summit (or free extremity) towards the base. Examination with a higher power (such as Zeiss, obj. F) does not, however, render this striation more plain, and I have not found it sufficiently distinct to be drawn. Thinking that this appearance was perhaps produced by intracellular fibrils such as Klein has described (*op. cit.*, p. 327), I treated portions of the thyroid gland of the Tortoise according to the three different methods recommended by the author of that paper (*ibid.*, pp. 319, 321, and 327), but without being able to see the intracellular reticulum distinctly. It must be borne in mind, however, that these cells are much smaller than those on which Klein made his observations.

Plate 68, fig. 8, shows an appearance probably connected with this striation. It is taken from the thyroid of a Kitten, injected by the method of puncture with silver, hardened in alcohol, and stained with hæmatoxylin. In addition to endothelial markings of the lymphatic, which are clearly seen (1, 1), on altering the focus the epithelial cells are seen studded with a quantity of dots and short lines, also stained with the silver, which I can only explain by supposing that the ends of intracellular fibrils have become stained by the silver, and are seen either endways or somewhat obliquely. The figure which Peremeschko gives (*op. cit.*, Plate 16, fig. 1) of a profile view of the processes at the base of the epithelial cells, showing minute dots arranged in parallel rows running in the longitudinal axis of the cells, suggests the idea that it is these structures which produce the appearance of dots and lines, stained with silver, just described, and also possibly that of the intracellular striation. My own observations on the existence or non-existence of processes at the base of the epithelial cells have not led to any definite results.

Reticulum.—On viewing from the surface the epithelium of the vesicle in the thyroid gland of the Tortoise hardened in chromic acid, a delicate network is seen running between the individual epithelial cells, which stains darkly with hæmatoxylin. (See Plate 68, fig. 5. In this figure the epithelial cells present in the meshes of the reticulum are not represented.) This reticulum, which is probably formed by coagulated intercellular substance, is seen to be thickened at intervals. When viewed somewhat obliquely the meshes of the reticulum appear to be made up of delicate lamellæ running between the epithelial cells, although it is difficult to speak with certainty on this point. In spirit, and in osmic acid preparations, this reticulum is also observed

in a surface view. In a profile view of the epithelium of the Tortoise (hardened in alcohol) there are seen at intervals amongst the epithelial cells, narrower cells with much elongated nuclei, which latter take the hæmatoxylin stain more darkly than the nuclei of the epithelial cells. On examining these cells carefully they are found to be expanded somewhat like a fan at their summits, and also sometimes at the base, although the latter is not always the case. Probably these are more or less branched cells, which are situated in the swollen parts of the reticulum above-mentioned. The reticulum as well as the *club-shaped cells* just described are very plainly seen in the thyroid glands of the Conger Eel and Skate. Plate 68, fig. 6, gives a surface view of the epithelium as seen in a section of the thyroid of the Conger Eel, hardened in spirit. Amongst the epithelial cells in this figure are seen the branched or stellate nuclei of the club-shaped cells. In this animal, as well as in the Skate, the club-shaped cells are often seen in great numbers in a profile view of the epithelium. It is a common thing to see two of these cells in close approximation to each other, almost suggesting the appearance of stomata opening into the cavity of the vesicle. That these cells however bear an important part in the absorption from, or secretion into, the cavity of the vesicle is, I think, probable, although I am unable to speak more definitely on this point. In the Conger Eel the expanded summits of the club-shaped cells may often be observed projecting slightly beyond the surface of the epithelial layer.

The reticulum is also seen in the thyroid gland of Mammals, such as the Sheep and Kitten.

Basement membrane.—In a double-injected gland of the Tortoise in which the blood-vessels were injected with carmine-gelatine and the lymphatics with Berlin-blue solution, I have found that the arteries are surrounded more or less completely by their accompanying lymphatics; the capillaries, on the contrary, run between the epithelium of the vesicles and the smallest ramifications of the lymphatics, which in this animal are found between almost all individual vesicles. This relation of the lymphatics to the capillaries and arteries can be easily seen in uninjected sections of the same gland, also in those in which the lymphatics only are injected by the puncture-method. Plate 68, fig. 9, exhibits this relation of the minute lymphatics to the capillaries in the thyroid of the Tortoise. It represents a transverse section of the walls of two adjacent vesicles with the intervesicular structures. 2, 2 are the cavities of the two vesicles, 1, 1 their epithelial walls. In the centre is seen the small lymphatic (3, 3), and between this and the epithelium on each side the capillaries (4, 4, 4). As the ramifications of the capillaries are much more minute than those of the lymphatics, spaces are frequently left between the capillaries in which the endothelial wall of the lymphatic is *not* separated from the epithelium by a blood-vessel, and at these points the epithelial cells appear usually somewhat elongated. Two of such points are seen in the figure (Plate 68, fig. 9). In osmic acid preparations of the Tortoise, where the wall of a vesicle happens to be cut obliquely, lying outside the epithelium, is seen the nucleated membrane formed by the endothelial cells of the

lymphatics, and on this are observed fine wavy striæ, which are evidently due to a delicate layer of connective tissue interposed between the endothelium and epithelium in which the capillary blood-vessels ramify (see Plate 68, fig. 7). Judging from the appearance presented by this layer of connective tissue when examined from the surface, it is probable that it forms a continuous layer even at points where the lymphatics are not separated from the epithelium by capillaries, although in vertical sections I have not been able to trace this layer at these points. I have not observed any other form of basement membrane, nor have I been able to detect by maceration in MÜLLER's fluid, any basement membrane resembling the "Drüsenkörbe" described by BOLL.

III. *Contents of Vesicles.*

Literature.—FREY and PEREMESCHKO describe the contents of the vesicles *in the embryo* as consisting of a finely granular substance in which cells and nuclei are embedded. PEREMESCHKO figures these cells and says that they differ from the epithelial cells by their round shape and *smaller* nuclei. He finds that in larger embryos vesicles filled with colloid masses are met with here and there. In young animals the greater part of the vesicles are filled with this mass, and finally in adult animals it is very rare to meet with vesicles without colloid.

KÖLLIKER speaks of the contents of the vesicles in the healthy human thyroid gland as a clear, yellowish, somewhat tenacious fluid containing a large amount of albumen. BOŹCHAT alludes to the cavities as being first partly filled with a fluid of viscid appearance in which float granules in more or less abundance. ZEISS describes the small vesicles as being first filled with a clear fluid, soluble in water, which is related to albuminous substances in its chemical behaviour, and considers that it is not essentially different from the colloid masses, which often accompany, and subsequently replace it entirely. Numerous ill-defined, granular disintegrating cells, remains of protoplasm and nuclei, are present in it, also fat granules, cholesterine and octohedral crystals of oxalate of lime.

Almost all observers allude to these contents of the vesicles being subsequently replaced partly or entirely by a substance to which the term of *colloid* is applied.

Colloid substance.—With slight variations this body is described as a homogeneous, transparent, more or less yellow substance (VIRCHOW, FREY, KÖLLIKER, BOŹCHAT), viscid in character (VIRCHOW, KÖLLIKER, BOŹCHAT), and completely filling the vesicles in the adult state (FREY). PEREMESCHKO finds that it usually fills the whole vesicle, but that in other cases it appears as smaller or larger transparent drops adhering to the wall of the vesicle; in still other cases it appears as smaller or larger transparent granules (sago-grains, ECKER) suspended in a finely granular mass. Amongst these granules (according to PEREMESCHKO) are found occasionally, even in adult animals, the above-named cells, and stages of transformation of these into the transparent colloid granules may be observed.

Under the microscope the colloid substance is either quite amorphous and homogeneous, or amorphous and slightly granular (VIRCHOW), or it encloses cells, or débris of cells, and granules (VIRCHOW, KÖLLIKER).

BOÉCHAT describes the appearance it presents when coagulated by re-agents, namely, that of hard masses of rounded shape with regular or jagged borders.

ZEISS observed true colloid masses in the smaller follicles, appearing in specimens prepared in MÜLLER's fluid as homogeneous, round, bean or egg-shaped masses, which are rarely granular, with a peculiar bluish or yellow tint, and often consisting of concentric layers. He found that these are at first suspended in the clear fluid, but soon accumulate layer upon layer on their surface, until they completely fill the vesicle.

The *colloid substance* is considered by HENLE and KÖLLIKER as pathological in character, while BOÉCHAT and PEREMESCHKO regard it as a physiological product. ZEISS, although agreeing with the last-named observer in having found it present in all Mammals examined, expresses no opinion either way. LEYDIG has also found it present in bony Fishes, Sharks and Rays, in Reptiles and Birds, and on that account he is *not* inclined to regard it as a pathological product when occurring in man.

Observations.

The following are the chief constituents of the contents of the vesicles which have been observed. They are considered entirely *from a morphological point of view*, and are described as they appear in specimens which have been *hardened in alcohol*, unless the contrary is stated. (Some of the specimens had been previously injected with Berlin blue or with nitrate of silver.)

(*a*) *Homogeneous* or *granular material* ("Colloid").
(*b*) *Red blood-corpuscles.*
(*c*) *Colourless blood-corpuscles.*
(*d*) *Rounded masses*, which stain darkly with hæmatoxylin or of a bright yellow colour with picrocarminate of ammonia.
(*e*) *Crystals* and *Pigment.*

(*a*) *Homogeneous or granular material.—Dog.*—It will be convenient to describe first more fully the appearance of this substance in the Dog, and afterwards to mention its characters in other animals. In my previous paper (*op. cit.*, p. 560) I described in the vesicles of the Dog's thyroid gland a peculiar material, which in hardened specimens shrinks away from the walls of the vesicle and forms a solid mass in the centre. In sections stained in picrocarminate of ammonia it appears as a finely granular substance, and stains of a more or less bright yellow hue. In specimens coloured with hæmatoxylin, on the other hand, it presents the appearance of an opaque, grey, or greyish-violet mass which is generally uniform, but sometimes finely granular in appearance. This substance was present in all the Dogs (10 animals) examined in

the present research. In the youngest Dog examined (5 weeks old) the vesicles were almost filled with a granular material, with smooth outline, which assumed a grey tint on staining with hæmatoxylin. In the remaining Dogs the quantity present varied very much. At one time the vesicles appeared completely filled with it, whereas at another time it formed a shrunken mass only occupying a small portion of the cavity. The intensity with which this material took the hæmatoxylin or picro-carminate stain also varied much in different glands. It may be conjectured that this variation in the amount present corresponded to different states of functional activity of the gland.

A substance apparently similar was found in greater or less quantity in the thyroids of all the other Mammals examined. It varied however greatly in amount.

Birds.—A similar substance was observed in the thyroid glands of the Fowl and Pigeon. The thyroid glands of two Rooks were examined; in one of these Birds the vesicles in both glands were found almost filled with a homogeneous material staining more or less with hæmatoxylin. In the second Rook, on the other hand, a very few vesicles contained this material, the great majority of them appearing either empty or much compressed.

In the thyroids of all the Tortoises examined (14) the vesicles were found almost or entirely filled with this homogeneous material.

In the Frog an apparently similar homogeneous material was invariably present. Its outline was usually much indented.

Fishes.—In the Conger Eel the vesicles contained a homogeneous material, often with indented edges. In hæmatoxylin specimens sometimes, whilst this material presented a yellow colour, its central portion, to a greater or less extent, was of a bright violet tint. Rounded masses staining more or less with hæmatoxylin were often found in the vesicles.

In the Skate in one instance the vesicles contained a comparatively small quantity of homogeneous material, which was much shrunk and contained numerous clear round spaces. In other glands the contents of the vesicles consisted of coarsely granular masses or of globules of various sizes.

From the description just given there can be no doubt, I think, that, in adult animals at least, the homogeneous or granular material corresponds with the "colloid substance" of authors, and from its almost universal occurrence in the gland-vesicles of adult animals, I have no hesitation in regarding it as a *normal* product of the organ.

When this homogeneous material shrinks away from the walls of the vesicle its outline often presents an indented appearance, as if beset with pellucid drops (see Plate 68, figs. 10 and 11, taken from the Tortoise, and Plate 68, fig. 4, from the Conger Eel). These have been considered as drops of colloid adhering to the walls of the vesicle (see above—Peremeschko). For my own part I regard this appearance merely as the result of contraction of the viscid material (colloid) contained in the

vesicle, which probably takes place in the following manner:—The vesicle being full of the viscid material is placed in the fresh state into alcohol. This coagulates the viscid material and causes it to shrink. When this takes place, however, it is necessary either that the walls of the vesicles should fall in, or that the space left between the epithelial wall and the coagulated content should be filled with some other fluid. It is probable that a clear fluid exudes from the epithelial cells in the form of a drop from each cell, producing the indentations above mentioned. This will also explain the occurrence of a pellucid appearance around the *large round cells* found in the viscid material in the Tortoise, which will be described further on (see Plate 68, fig. 11). For in the latter case when the viscid material contracts, fluid probably exudes from the large round cells, and as it escapes equally all round, in the case of one of these cells lying singly, the pellucid appearance is more or less spherical in shape, as shown in the figure.

I have seen no sign of this homogeneous substance consisting of "concentric layers," or of its accumulating "layer upon layer," as described by Zeiss.

(*b*) *Red blood-corpuscles.*—Red blood-corpuscles are not uncommonly seen in the vesicles of the thyroid gland. They have been observed in the glands of Dogs (whose age ranged from 5 weeks to 7 or 8 years), in the glands of several Tortoises, and in the Conger Eel. The blood-corpuscles, which from their being situate *in* the homogeneous material (colloid) above described had beyond all doubt entered the vesicles *during life*, are sometimes few in number, but at other times they *completely fill the vesicle*. They either appear collected (or fused) into a ball in the centre, or are scattered throughout the contents of the vesicle (see Plate 68, fig. 10, from the Tortoise). They are also frequently met with arranged in a layer close to the epithelial cells (see Plate 68, fig. 12, from the Dog). The corpuscles are observed *in different stages of disintegration and decolorisation*. In some instances in the thyroid gland of the Dog the epithelial wall of a vesicle containing red blood-corpuscles was seen to be studded with a quantity of *minute yellow granules*, no doubt due to the absorption of the colouring matter of the escaped red blood-corpuscles by the epithelial cells or inter-cellular reticulum. (Plate 69, fig. 13, *a*, *a*, *a*, *a* shows these granules in the epithelial wall of two vesicles containing red blood-corpuscles.) Colourless blood-corpuscles are sometimes observed in the cavity of the vesicles mixed with the coloured ones.

In the ten Dogs examined, whose age varied from 5 weeks to 12 years, one or more vesicles containing red blood-corpuscles were found, with one exception, in *all* instances either in one or both glands. (These glands were all *uninjected*.) The exception was that of a Dog (female, aged 12 years) in which appearances rendered it very probable that there had been an escape of red blood-corpuscles, but this could not be ascertained for certain.

In one gland of a Dog from this series (female, aged 7 or 8 years) *a very large proportion of the vesicles* contained red blood-corpuscles in greater or less number (see

Plate 68, fig. 12). The opposite gland of this Dog also contained numerous vesicles with red blood-corpuscles. In the gland of another Dog (male, nearly 2 years old) numerous vesicles containing red blood-corpuscles were also present. (In these two Dogs the red blood-corpuscles were more disintegrated and less distinct than in many others, but from a study of the changes which the red blood-corpuscles undergo in the vesicles, I think there can be no doubt of these being of this nature.) From these observations it appears that *in the Dog, at least, an escape of red blood-corpuscles is almost constantly taking place into a greater or less number of vesicles.* As only a few sections from each gland were examined, it is obvious that the number of vesicles containing red blood-corpuscles in all probability greatly exceeded that actually observed. The frequency of the escape of red blood-corpuscles into the vesicles of the Dog, together with its occurrence in the glands of such different animals as the Tortoise and Conger Eel, I think, renders it *very probable that the passage of red blood-corpuscles into the vesicles is a normal occurrence in the thyroid gland.*

In the thyroid gland of a Seal (*Phoca vitulina*) which I obtained through the kindness of the late Professor A. H. Garrod, F.R.S., the epithelium of *almost all* the vesicles contained numerous dark red granules, which on close examination appeared to be minute crystals. Arguing from the appearance above-described in the epithelium of the Dog, I attributed these to the absorption of colouring matter from blood effused into the cavity of the vesicles. Subsequently also on careful examination I found that *a large proportion, in some parts certainly the majority, of the vesicles* contained structures which were very probably red blood-corpuscles, but, perhaps owing to the gland not being quite fresh, they could not be certainly identified with these. In the thyroid gland of the Rook, also, in one instance, yellow granules, some of considerable size, were seen in the epithelial wall of *almost all the vesicles of both glands* (see Plate 69, fig. 14), and yellow masses were found in the cavities of some of the vesicles which I was inclined to consider as fused red blood-corpuscles. A careful examination of these glands led me to believe that in this Bird *an extensive hæmorrhage* had taken place into the vesicles, and that the Bird was killed just as the colouring matter of the blood was being reabsorbed.

It is possible, of course, that the glands of the Rook and the Seal just mentioned, as well as those of the two Dogs (in which an extensive escape of red blood-corpuscles was observed), may be pathological in character, in which case considerable interest would also obviously attach to them. These observations, however, I think at least render it *very probable that normally, under certain circumstances, an escape of red blood-corpuscles takes place into a large proportion of the vesicles of the thyroid gland;* they certainly encourage to further research in this direction with a view, if possible, of throwing light on the function of the organ.

The blood which thus escapes into the vesicles contributes, no doubt, *to a large extent* to the formation of the "colloid" material which they contain, and it seems highly probable that this escape of red blood-corpuscles into the vesicles, with their

subsequent disintegration, has an important bearing both on the physiology and pathology of the gland.*

I may mention that vesicles containing many red blood-corpuscles have also been observed in the thyroid gland of Man (male aged 4 years, and adult Man).

The red blood-corpuscles in the vesicles have probably hitherto escaped detection in the physiological state of the gland, owing to the rapidity with which they become melted down, as it were, and thus rendered invisible. From the fact also that the red blood-corpuscles in the *vessels* in these specimens often present no distinct structural characters, it is quite possible that red blood-corpuscles may be present in the *vesicles* at the time of death, although they are not distinguishable as such in sections of the hardened gland.

Although the following applies to a pathological state of the gland it is interesting, as showing that what I have described in the *normal* state has been already observed in abnormal conditions. In speaking of *vascular goitre* ("Gefässkropf") KÖLLIKER (*op. cit.*, p. 482) says that in this disease, besides a hyperæmic condition, there are numerous aneurysmal dilatations of the small blood-vessels which ECKER regards as arteries and coarse capillaries. By the bursting of these dilatations apoplectic vesicles of different sizes are formed, which may become modified in various manners by the blood undergoing different changes, &c.

(*c*) *Colourless blood-corpuscles.*—On examining a section of the thyroid gland of the Tortoise, it is common to find in a large proportion of the vesicles, situate in the homogeneous material, a greater or less number of *large round cells* (see Plate 68, fig. 11). These cells when perfect are round in shape, and present a granular cell-substance. They are each provided with a single, round, or oval-shaped nucleus. Both the cells and nuclei are *larger* than those of the epithelium of the vesicle. In this respect they differ from the cells described in the vesicles by PERMESCHKO (*vide supra*, p. 589). Their number in each vesicle often appears to be considerable, for in a section, which of course only includes a portion of the contents of a vesicle, it is not uncommon to find half-a-dozen or more of them. As they have been found in greater or less number in all the glands of the Tortoise, which have been sufficiently well prepared, I conclude that their presence in the cavities of the gland-vesicles is a normal phenomenon in this animal.

It was first thought that these cells might be parenchymatous cells, such as I have

* Amongst other suggestions which occur to one in this connexion, it is impossible to avoid conjecturing, whether rightly or wrongly, that the *anæmia* which so commonly accompanies certain forms of enlargement of the thyroid gland (*goitre*) may be due to an excessive destruction of red blood-corpuscles in the manner above-described [see, for instance, ERICHSEN, 'Science and Art of Surgery,' 5th ed., vol. ii., p. 297: "There is a remarkable connexion between tumors of the thyroid gland of this kind (simple hypertrophy) and a general anæmic condition of the system. In London nothing is more common to find than a certain degree of bronchocele in pale and bloodless women and girls; indeed, so frequent is the coincidence that it is impossible not to regard it in the light of cause and effect"].

already described as passing into the vesicles of the Dog's thyroid gland; but from a comparison of them with the fresh blood of the Tortoise, and from the occurrence of very similar cells in the blood-vessels in sections of the hardened gland of the same animal, I think there can be no doubt that they are *colourless blood-corpuscles*. It is scarcely necessary to mention that these cells can be easily distinguished from any *post-mortem* falling in of the epithelial cells; moreover, in these specimens the epithelium was usually intact. In specimens prepared in osmic acid these cells come out very clearly, and their cell-substance presents a coarsely-granular appearance.

Some of these Tortoises were killed by decapitation, after a ligature had been tightly tied round the neck, but the method of killing could have had no effect in causing the escape of these cells, for they were found equally in those killed by simple decapitation without ligature. The appearances observed also left no doubt that they had entered the vesicles during the life of the animal.

A few large nucleated cells have been observed in the vesicles of the thyroid gland of the Rook, which were probably colourless blood-corpuscles.

In one thyroid gland of a Dog (aged 9 weeks), many vesicles contained a quantity of cells with granular cell-substance and indistinct outline, both in their cavity and in the substance of the epithelial wall. The nuclei of these cells were smaller, and stained more deeply with hæmatoxylin than those of the epithelial cells. These cells, which I conclude were colourless blood-corpuscles, I am inclined to regard as a pathological appearance, as they were only seen in part of one gland of this Puppy. Their occurrence in the vesicles cannot therefore be compared with the migration of colourless corpuscles into the vesicles of the Tortoise, which from its being constantly present is doubtless a normal phenomenon.

It appears, therefore, that *a migration of colourless blood-corpuscles into the vesicles is a physiological occurrence in the thyroid gland of the Tortoise, but that it may also occur in other animals.*

(*d*) *Rounded masses*, embedded in the homogeneous material, have been observed in various animals. These are usually homogeneous in character, with smooth, or sometimes jagged outline, and stain darkly with hæmatoxylin (more so than the surrounding homogeneous substance), or of a bright yellow colour with picrocarminate of ammonia. They have been observed in the thyroids of Dog, Rook, Fowl, Pigeon, Conger Eel, and Skate. In the latter they sometimes presented the following appearance. The central portion was deeply stained with hæmatoxylin, whilst the periphery remained almost colourless. Whether these are pathological in character or not, and what relation they bear to the homogeneous contents of the vesicle, I am unable to say.

(*e*) *Crystals* were found in the homogeneous material in the gland-vesicles of the Tortoise and Rook, the latter being the bird in which it was concluded (as above-described, p. 593) that hœmorrhage had taken place into almost all the vesicles. The crystals in this case were present in considerable numbers.

Pigment.—Masses of brown pigment-granules were frequently seen in the homogeneous material in the thyroid glands of the Frog. They are probably the result of an escape of red blood-corpuscles into the vesicles, but this point could not be ascertained for certain. Numerous brown pigment granules were also sometimes seen in the homogeneous material in the vesicles of the Conger Eel.

IV. *Parenchyma.*

In my previous paper on this subject (*op. cit.*, p. 563) I described and figured, as normal structures in the thyroid gland of the Dog, some large round cells provided with a single oval-shaped nucleus, to which I applied the term "parenchymatous cells." From the different appearances they present and their different positions in regard to the wall of the vesicle, I concluded that a migration of these cells takes place into the cavity of the vesicles by displacement and compression of the epithelial cells.

Zeiss (the only observer, as far as I am aware, who has since written on the subject) has been unable to observe any such migration of cells into the vesicles. In order to see these cells well, it is of course necessary that the sections should be very thin.

To the description of these cells in the Dog, as given in the above paper, I have little to add. I have since also observed them in the glands of young Dogs (aged 5 and 9 weeks respectively). Whilst still maintaining the view that these cells migrate into the cavity of the vesicles, I think it is quite possible that some of them may originate between the epithelium of a vesicle and the capillaries, and from thence pass by compression or separation of the epithelial cells into the vesicle.

In the thyroid gland of the Cat, parenchymatous cells are also present in considerable numbers, although not nearly so numerous as in the Dog. They resemble very much those seen in the latter animal, and appear to migrate into the vesicles in a similar manner. A few cells, apparently of similar character, have also been seen in the thyroid of the Rabbit.

In the thyroid of the Pigeon large groups are frequently seen, consisting of cells, which are larger than the adjacent epithelial cells (see Plate 69, fig. 15). They are round or oval in shape, and are provided each with a single spherical or oval-shaped nucleus. They resemble very much the parenchymatous cells seen in the Dog; but whether they, like them, migrate into the vesicles I am unable to say, as I have not observed any of them in the substance of the epithelial walls.

Round cells were sometimes seen in the epithelium of the Skate, but the nature of these could not be ascertained. In the Conger Eel groups of cells somewhat rounded in form were found amongst the cylindrical epithelial cells, but in all probability these were connected with the origin of the small vesicles in the walls of larger ones, as above described.

V. *Lymphatics and their contents.*

Literature.—Lymphatics.—I have already described (*op. cit.*, p. 559) and figured in the thyroid gland of the Dog a dense system of lymphatics consisting of lymphatic vessels, spaces, and canals, traversing the gland in all directions. In opposition to Frey (quoted by Henle, *loc. cit.**), who I supposed was the last observer who had then studied these structures, I laid stress on their not ending in blind extremities, as described by that author, and on their being much more numerous than he supposed. After the publication of my observations I became aware that in 1873 P. A. Boéchat (*op. cit.*) had described a very similar network of lymphatics in this organ. Boéchat (p. 39) found that the lymphatics of the thyroid gland of the Dog form a vast hollow network, contained in the stroma of connective-tissue, which serves as framework of the organ. The network is formed of lymphatic sinuses, which communicate with one another largely. In its meshes are contained the thyroid cavities (vesicles) whose walls are at many points adherent to those of the lymphatics. He studied them in specimens injected with nitrate of silver and with Prussian blue. Peremeschko (*op. cit.*) had also described in the thyroid of the Dog a rich network of lymphatics, which surrounds small groups of vesicles; also large lymphatic canals, which frequently surround the arteries like sheaths, and under the capsule a network of lymphatic spaces ("Gänge").

Since the appearance of my paper Zeiss (*op. cit.*, 1877) has given a description of these vessels, which agrees on the whole with that given by Boéchat and myself. He finds that the lymphatics form wide cavernous canals, which are not confined to the larger septa, but also as minute clefts (always lined by the same endothelium) encircle the follicles separately or in small groups of two, four, or six. As regards the endothelium, he finds its cells elongated and with wavy borders, the margins of neighbouring cells either fitting accurately into one another or else leaving between them small round or oval areas of cement-substance, which become more darkly stained than the gelatine (employed with the nitrate of silver in injecting) and project somewhat above the level of the endothelial cells. The lymphatic spaces frequently surround the arteries and veins for considerable distances.

Contents of lymphatics.—In my former paper (p. 562) I described the appearances presented by the contents of the lymphatics, namely, that of a homogeneous or granular mass, and argued in favour of its being during life of a viscid consistence. I also drew attention to the close resemblance (amounting, as I considered, to morphological identity) between the coagulated contents of the lymphatics and those of the vesicles, and urged its importance as tending to show that the material which is formed in the vesicles is carried off by the lymphatics. Zeiss has also noticed the coagulability of the contents of the lymphatics, but whilst admitting that they stain in similar

* I have not been able to obtain the original article of Frey in the "Viertoljahrsschrift der Naturforschenden Gesellschaft in Zürich," viii., i., 1863.

manner with colouring agents, he points out that the contents of the lymphatics show an entire absence of the phenomena of contraction observed in the contents of the vesicles.

Observations.

Mammalia.—A system of lymphatics, resembling in its general characters that described in the Dog, has been observed in the thyroid glands of other animals (*e.g.*, Kitten, Horse, Rabbit) by means of injections by the method of puncture. The extent of distribution of the network and the minuteness of its ramifications appear to vary in different animals.

The homogeneous or granular contents of the lymphatics have been observed in the same vessels in the thyroids of other animals (*e.g.*, Horse, Sheep). This material occurs both in injected and in uninjected specimens. The quantity varies much in different glands of the same species of animal, it being sometimes present in large quantities and at other times being apparently entirely absent. With a view of ascertaining whether the occurrence of this material in the lymphatics was peculiar to either sex, or to any particular age, I examined the thyroids of Dogs of various ages in both sexes, but, as might be expected, without any definite result. It appeared to be present or absent regardless of the sex or age of the animal. The margins of the coagulated content of the lymphatics undoubtedly do not present the indented appearance seen in the vesicular contents, but this is, I imagine, simply due to the absence of epithelial cells, from which drops of clear fluid could exude (see above, pages 591 and 592).

Aves.—In the thyroid gland of Birds, on the contrary, *I have been unable to inject any system of lymphatics.* My observations on this point are as follows:—On injecting the thyroid gland of a Pigeon by the method of puncture with Berlin blue, it swells up, and the injection is seen running *in the jugular vein* with which the gland is in close apposition. (It will be remembered that on injecting the Dog's thyroid gland in a similar manner, the injection was seen emerging from the gland *in lymphatic vessels*, which ran to neighbouring lymphatic glands.) Examination of sections of the Pigeon's gland, thus injected, shows that the injecting fluid has entered vessels containing red blood-corpuscles. The same vessels become filled on injecting in like manner with a solution of nitrate of silver. To ascertain whether by the puncture-method the blood-vessels really become injected, another gland was injected with Berlin blue from the lower part of the carotid artery, the artery at the same time being secured above the gland. In this case, in which the injection had entered the capillaries and veins, it was evident that these were the same system of vessels that had been injected by the method of puncture. It may therefore be stated that on repeated injections of the thyroid gland of the Pigeon, both with Berlin blue and with nitrate of silver, by the method of puncture, *I have been unable to inject any system of lymphatic vessels, but have always found the injection in the blood-vessels of the gland.* The *blood-vessels* of the thyroid gland of the Pigeon also present the following pecu-

liarities:—The capillaries, as usual, form a network running between the individual gland-vesicles, but in proportion to the size of the vesicles they do not appear so minute or to have such complicated ramifications as in the case of other animals (*e.g.*, Sheep or Tortoise), but resemble more in their distribution the lymphatics in some of the Mammalia (see Plate 69, fig. 17). The veins frequently surround either partially or entirely the arteries which they accompany (see Plate 69, fig. 18). Immediately under the surface of the capsule numerous large veins are seen, and in the fibrous capsule itself layers of red blood-corpuscles have been observed, which appear to be contained in blood-vessels communicating with the veins in the interior.

In a preliminary communication presented to the Society on this subject (Proc. Royal Society, No. 185, 1878), I mentioned that once or twice I had noticed in the large veins on the surface of the gland (under the capsule) in addition to red blood-corpuscles, and perhaps coloured injection, a greater or less quantity of a material of homogeneous aspect, presenting an appearance similar to the material seen in the vesicles, also to that described above in the lymphatics of the Dog. I do not, however, attach any importance to this, as the contents of the blood-vessels in the Dog, when coagulated, also sometimes present this homogeneous aspect.

In the thyroid gland of the Rook no system of lymphatics becomes injected by the method of puncture, but the blood-vessels, presenting an appearance very similar to those in the Pigeon, become filled.

Reptilia.—In the thyroid gland of the Tortoise a network of lymphatics can be injected by the method of puncture, of which the smaller ramifications run between almost all individual vesicles. In specimens injected with nitrate of silver the lymphatics are seen to be lined by a layer of endothelium, of which the cells are usually elongated and have a sinuous outline. The areas, stained dark with nitrate of silver (as described by Zeiss; see above, p. 595), are well seen in this animal (see Plate 69, fig. 16). They vary much in size and shape, and often appear to have a double contour. Usually, though not always, they are situate on the boundary line separating two endothelial cells. In the Tortoise the lymphatics have not usually any visible contents, but they sometimes contain a granular material which stains scarcely at all with hæmatoxylin.

Pisces.—In the thyroid gland of both Skate and Conger Eel an extensive system of vessels lined with endothelium becomes injected by the method of puncture. (A considerable portion of the injection, however, also passes into the meshes of the connective-tissue.) Being doubtful as to whether the vessels thus injected were of lymphatic or blood-vascular character, I injected the thyroid gland of a large Conger Eel with solution of Berlin blue from the blood-vessels (dorsal aorta and efferent branchial trunks); and from a comparison of the vessels injected in this case with those filled on injection by the puncture-method, I have been led to the conclusion that in the thyroid gland of the Conger Eel at least, as far as my observations extend, *there is no evidence of any system of lymphatic vessels.* Large veins, filled with red

blood-corpuscles, are frequently seen in this thyroid gland amongst the vesicles, and between these veins and the epithelial wall of the vesicles capillaries often ramify.

VI. *Blood-vessels.*

Literature.—The great richness of the thyroid gland in blood-vessels and the abundance of its capillaries, which form extensive ramifications on the surface of the vesicles, are generally acknowledged.

Zeiss (*op. cit.*, p. 34) describes, in specimens in which the blood-vessels are injected with Berlin blue, knotty dilatations forming dark blue drops situate laterally on the capillaries, or similar structures forming apparently the blind end of a capillary. He observed these in the Rabbit, Cat, and Rat, and says he was unable to observe these peculiar bulging or drop-like dilatations in uninjected specimens, although he speaks elsewhere of the capillaries in uninjected specimens being wedged in between the epithelial cells, and forming small projections towards the epithelial layer. Zeiss considers that the presence in such great numbers of these nodules or dilatations on the capillaries may retard to a considerable extent the flow of blood through the vessels of the gland, thereby favouring any secretion from, or absorption into, the blood-vessels, which may take place either in connection with the vesicles or with the lymphatics. Neither Boéchat nor Peremeschko make any mention of these dilatations, and in the figure which the latter observer gives of the blood-vessels no sign of them is visible.

Observations.—In the thyroid gland of the Tortoise I have frequently observed, in transverse sections of the epithelial walls of the vesicles (in uninjected specimens), projections of the capillaries between the epithelial cells, towards the interior of the vesicles. I have endeavoured repeatedly to show these by injections of the blood-vessels of the gland in this animal, but without much success, owing to a shrinking of the epithelial cells. It appears very probable that it is through these projections of the capillaries that the escape of blood-corpuscles (red or colourless) takes place into the cavity of the vesicles. In fact, in one specimen I could see in a vesicle, close to one of the capillary projections, some red and colourless blood-corpuscles which had apparently just escaped from a blood-vessel through this channel. Some peculiarities of the blood-vessels in the thyroid glands of Birds have been alluded to under the head of lymphatics (*vide supra*, p. 598), and those of the Conger Eel have also been referred to above (p. 599).

The relation of both large and capillary blood-vessels to the lymphatics has been discussed in considering the basement membrane (*vide supra*, p. 588).

VII. *Undeveloped portions.*

In the thyroid gland of the Dog bodies of considerable size are frequently seen, which differ entirely in structure from the rest of the gland (see Plate 69, fig. 19). They are rounded or flattened in shape, usually situate on the surface of the organ, and

possess the following structure (see Plate 69, fig. 20). They consist of a solid mass of more or less *cylindrical rows of cells* (Plate 69, fig. 20 1, 1, 1) which are much convoluted and interlace in all directions. Between them run capillary blood-vessels and also probably some lymphatics. These "cylinders" are composed of cells resembling epithelial cells, columnar or cubical in shape, those on the surface of the cylinder, next to the capillaries, being arranged at right angles to those vessels (Plate 69, fig. 20, 2). Each cell is provided with a nucleus usually oval in shape. In very few, if in any, of these cylinders have I been able to detect any central canal. In Dogs aged three months and upwards I have usually observed these "undeveloped portions" as distinct bodies, not continuous with the normal gland-tissue, but separated from it by layers of connective tissue, and frequently lying in depressions on the surface of the gland (see Plate 69, fig. 19). They appear to be portions of gland whose development has become arrested at an early stage, and there was in these Dogs, usually, no evidence to show that these bodies were undergoing any further development. Exceptions to this, however, occur; for example, in a male Dog (aged nearly 2 years) an "undeveloped portion" was found which was not distinctly separated from the rest of the gland-tissue, and contained well-formed vesicles showing that it was undergoing further development.

In the Kitten similar undeveloped portions are seen, which are sometimes observed to be continuous with the ordinary gland-tissue. In this case a formation of vesicles from the cylinders of cells appears to be taking place by the growth into them laterally of processes of connective-tissue with blood-vessels, and by their excavation into vesicles. In the Kitten the cylinders are less convoluted than in the Dog, and throughout the gland the fully-formed vesicles frequently appear grouped in rows, which have a more or less parallel arrangement.

"Undeveloped portions" have also been observed in the thyroid glands of the Sheep, Seal, and Rook. In the Seal there was a large quantity of fibrous tissue between the cylinders, as well as between the vesicles in the ordinary gland-tissue.

Somewhat similar, but much shorter, cylinders of cells have been seen in the thyroid gland of the Pigeon scattered throughout the gland.

There can be no doubt that these bodies are portions of the gland of which the development has progressed no further than the stage described by W. Müller (*vide supra*, p. 583), in which the gland consists of a network of cylindrical tubes, at first irregular, but afterwards becoming more regular in character; the tubes (according to Müller) being composed of short cylindrical epithelial cells, placed radially on the surface of the envelope and being at first *solid*, but subsequently provided with a lumen. I have also observed these cylinders in the thyroid glands of fœtal Pigs (measuring about 2½ inches in length). The occurrence of these undeveloped portions continuous with the ordinary gland-tissue in the thyroids of *young* animals of course merely indicates that a formation of gland-tissue is taking place in the manner described by W. Müller.

In the thyroid gland of the Dog, however, it is remarkable that although these bodies are of frequent occurrence, there is, usually, in Dogs aged three months and upwards no evidence to show that they are undergoing further development. It is an interesting question whether the undeveloped portions, which are distinct from the gland, may not under certain circumstances become developed into true gland-tissue, and either give rise to the isolated lobes mentioned by various authors, or simply cause an increase in size of the gland itself.

Masses of *lymphoid tissue* have been observed in the thyroid glands of Kitten and Pigeon.

GENERAL CONCLUSIONS.

1. In the Frog, in opposition to the statements of several observers, the thyroid gland forms two small bodies situate, one on either side, on the ventral surface of the hyoid cartilage (or bone) usually but not always in direct contact with this structure. They are placed either between the hyoid cartilage and the hyoglossus muscles, or else immediately to the outer side of the latter, just anterior to the point of their convergence.

2. In the thyroid gland of adult animals the *vesicles* form closed cavities which are not in communication with one another to any extent. Branched vesicles, which may be present in the glands of adult animals, probably merely indicate that an increase in the number of vesicles is taking place by the growth into them of involutions of their walls. In the Conger Eel small vesicles are frequently observed in the walls of larger ones, into which they appear sometimes to burst by causing a flattening of the epithelial walls.

3. Whilst the shape of the *epithelial cells* varies much in different animals, the general tendency of these cells throughout the Vertebrate series, as far as examined, is to the columnar form.

A delicate *reticulum* (as described by Zeiss) is found amongst the epithelial cells of the vesicles. In a profile view of the epithelium *club-shaped cells* are also often seen, which are probably more or less branched cells situate in the swollen parts of the reticulum.

The only form of *basement membrane* observed has been a delicate layer of connective tissue between the epithelium and the endothelium of the lymphatics in which the blood-vessels ramify.

4. The following are the chief *contents of the vesicles* which have been observed in the microscopical examination of specimens hardened in alcohol.

(*a*) *Homogeneous or granular material* ("colloid substance" of authors) has been observed with slight variations in all classes of the Vertebrata, and is undoubtedly a normal product of the organ.

(*b*) *Red blood-corpuscles* are frequently found in the vesicles in greater or less number and in various stages of disintegration and decolorisation.

In the epithelial wall of vesicles containing red blood-corpuscles, granules of yellow

pigment are sometimes seen, which appear to be due to the absorption by the epithelial cells or reticulum of the colouring matter of the effused blood. Observations on the Dog lead to the inference that in this animal *an escape of red blood-corpuscles is almost constantly taking place into a greater or less number of vesicles*, and its occurrence in other animals also (Man, Tortoise, Conger Eel) renders it *very probable that the passage of coloured blood-corpuscles into the vesicles is a normal occurrence in the thyroid gland*. Further observations (Dog, Rook, Seal) also render it very probable that *normally under certain circumstances an escape of red blood-corpuscles takes place into a large proportion of the vesicles of the thyroid gland*. The blood thus effused contributes to a large extent to the formation of the "colloid" substance, and probably has an important bearing on the physiology and pathology of the organ.

(*c*) *Colourless blood-corpuscles* are frequently found in the homogeneous material in the gland-vesicles of the Tortoise. A migration of colourless corpuscles into the vesicles is a *normal* occurrence in the Tortoise, but it may also take place in other animals.

(*d*) *Rounded masses*, staining darkly with hæmatoxylin or of a bright yellow colour with picrocarminate of ammonia, are seen in the vesicles of many animals. *Crystals* sometimes occur in the gland vesicles (Tortoise and Rook) and brown *pigment granules* have been observed in the colloid material in the vesicles (Frog and Conger Eel).

5. *Parenchymatous cells*, already described in the Dog, also occur in the glands of young Dogs and in less numbers in the Cat, in which animal they appear to migrate into the vesicles in the same manner as in the Dog. Large groups of round or oval-shaped cells, provided with a single nucleus, frequently occur in the thyroid gland of the Pigeon.

6. A network of *lymphatics* resembling in its general characters that previously described in the Dog, though differing apparently in the extent of its ramifications, is also seen in some other Mammals. The homogeneous or granular contents of the lymphatics in the Dog occur in both injected and in uninjected specimens and in very variable quantity; the amount being apparently independent of the sex or age of the animal.

In the thyroid gland of Birds (Pigeon) no system of lymphatics is injected by the method of puncture, but the injecting fluid enters the blood-vessels of the gland, escaping by the jugular vein.

The thyroid gland of the Tortoise contains a network of lymphatics, of which the smaller ramifications run between almost all individual vesicles. The endothelium of these lymphatics presents well marked areas of irregular size and shape, stained darkly with nitrate of silver. In the thyroid gland of the Conger Eel no evidence of a system of lymphatics was obtained by injections.

7. The *arteries* are ensheathed more or less completely in their accompanying lymphatics (Dog, Tortoise); the *capillaries*, on the contrary, run between the epithelium of the vesicles and the smallest ramifications of the lymphatics (Tortoise) leaving

intervals between them, where the lymphatic is not separated from the epithelium by blood-vessels. Projections of the capillaries between the epithelial cells towards the cavity of the vesicles occur frequently in the Tortoise. In the Pigeon the inter-vesicular ramifications of the blood-vessels are not so complicated or minute as in other animals (Sheep, Tortoise) but resemble more the distribution of lymphatics in some Mammals. The veins, in the Pigeon, frequently surround either partially or entirely the arteries which they accompany.

8. Bodies consisting of portions of the gland of which the development has proceeded no further than the stage of convoluted "cylinders of cells" described by W. MÜLLER are frequently seen in the thyroid gland. The cylinders are composed of cubical or columnar cells, resembling epithelial cells, those on the surface of the cylinder being arranged at right angles to the capillaries, which ramify between the cylinders. In Dogs aged three months and upwards these "undeveloped portions" usually form distinct bodies separated by layers of connective tissue from the rest of the gland. In young animals, such as the Kitten, they are often continuous with the ordinary gland-tissue into which they are evidently developing.

In conclusion, I wish to offer my best thanks to Dr. KLEIN for his direction in this research, which has extended over a period of several years.

EXPLANATION OF THE PLATES.

PLATE 68.

Fig. 1. Transverse section of a portion of the neck of a moderate sized Frog, on a level immediately anterior to the convergence of the hyoglossus muscles. (VÉRICK'S obj. O, oc. I. All the lenses employed in these drawings were VÉRICK'S, except an obj. F of ZEISS.)

1. Hyoid cartilage.
2, 2. Thyroid glands, situate directly on the ventral surface of the hyoid cartilage.
3. Hyoglossus muscles seen in transverse section.

Fig. 2. Outline of a branched vesicle from a longitudinal section of the thyroid gland of a Puppy (aged 5 weeks) hardened in alcohol. The section from which this was taken was purposely very thick, and the different portions of the vesicle, which are situate on different planes, can only be brought into view by repeatedly altering the focus. The epithelium is semi-diagrammatic. (Obj. F, oc. I.)

Fig. 3. From a transverse section of the thyroid gland of a Dog (aged 3 months) hardened in alcohol. Showing the outline of a branched gland-vesicle. This body appears at first sight to be composed of three distinct vesicles, but on focussing the continuity of these three can be traced at 1, 1. (Obj. 6, oc. I.)

Fig. 4. Transverse section of a portion of the wall of a large gland-vesicle from the thyroid gland of a Conger Eel, hardened in alcohol and stained with hæmatoxylin. (Obj. 6, oc. I., tube half drawn out.)

1. Cavity of the large vesicle.

2, 2. Its epithelial wall, of which the cells are represented semi-diagramatically.

3, 3. Small vesicles, which are situate in the epithelial wall of the large vesicle and are separated by much flattened epithelial cells from the cavity of the latter.

4. Another vesicle, separated by cells less flattened from the cavity of the large vesicle.

5, 5. Vesicles distinctly outside the wall of the large vesicle, which appear to be growing outwards.

The homogeneous contents of the small vesicles are seen with their jagged outlines. The vesicles represented in this drawing are not all quite on the same level.

Fig. 5. From a section of the thyroid gland of the Tortoise, hardened in a 1 per cent. solution of chromic acid (17 hours) and subsequently in diluted and in strong alcohol, and stained with hæmatoxylin. The intercellular reticulum is seen from the surface, but the intervening epithelial cells are not represented. (Obj. F, oc. I.)

Fig. 6. From a section of the thyroid gland of the Conger Eel, hardened in alcohol and stained with hæmatoxylin, giving a surface view of the epithelial cells amongst which are seen the branched nuclei of the club-shaped cells. They are probably connected with the intercellular reticulum. (Obj. 6, oc. I., tube half drawn out.)

Fig. 7. From a section of the thyroid gland of the Tortoise prepared as follows: injected by the method of puncture with $\frac{1}{4}$th per cent. solution of osmic acid, hardened in a 1 per cent. solution of the same acid for five hours, and subsequently in alcohol, mounted in glycerine. It shows (at 1) the epithelial wall of a vesicle cut obliquely, beyond which there projects (at 2) a membrane having delicate wavy striæ on it. (Obj. F, oc. I.)

Fig. 8. From a longitudinal section of the thyroid gland of a Kitten injected with nitrate of silver by the method of puncture, hardened in alcohol and stained with hæmatoxylin. The drawing represents a portion of the wall of a vesicle seen from the surface. By focussing from the interior of the vesicle, three layers can be distinguished. First, the oval-shaped nuclei, at regular intervals, stained with hæmatoxylin; secondly, a layer of dots and short striæ, stained with nitrate of silver, only represented in the centre of the drawing; and thirdly, the outlines of the endothelial cells of a lymphatic, also stained with silver (1, 1). (Obj. F, oc. II.)

Fig. 9. From a section of the thyroid gland of the Tortoise (injected with Berlin

blue by the method of puncture, but the injection had not penetrated into this branch of the lymphatics), hardened in alcohol and stained with hæmatoxylin, showing a portion of the walls of two adjacent vesicles with the intervening structures seen in transverse section. (Obj. F, oc. I.)

1, 1. Epithelial walls of two vesicles.

2, 2. Cavities of ditto.

3, 3. Small branch of lymphatic running between the two vesicles. The nuclei of some of its endothelial cells are seen.

4, 4, 4. Capillaries situate between the lymphatic and the epithelial cells.

This drawing also shows certain points at which the lymphatics are *not* separated from the epithelial cells by capillaries.

Fig. 10. From a section of the thyroid gland of the Tortoise, uninjected, hardened in alcohol and stained with hæmatoxylin. Section of a vesicle containing homogeneous material, with indented edges, embedded in which are numerous red blood-corpuscles partly separate and partly fused together to form a ball in the centre of the vesicle. Epithelium drawn semi-diagrammatically. (Obj. F, oc. I.)

Fig. 11. Section of another vesicle from the same thyroid gland as the preceding, prepared in like manner. The vesicle contains, embedded in the homogeneous material, several large round nucleated cells (colourless blood-corpuscles). Epithelium semi-diagrammatic. (Obj. 8, oc. I.)

Fig. 12. From a longitudinal section of the thyroid gland of a Dog (female, aged 7 or 8 years), uninjected, hardened in alcohol and stained with picro-carminate of ammonia. Numerous vesicles are seen in section, almost all of which contain in addition to the usual homogeneous material a greater or less number of partly disintegrated red blood-corpuscles. The epithelium, which is somewhat flattened, probably from the effect of the re-agents, is represented diagrammatically. (Obj. 6, ocs. I. and III.)

PLATE 69.

Fig. 13. From a transverse section of the thyroid gland of a Dog (male, aged 3 months), uninjected, hardened in alcohol and stained with hæmatoxylin. (Obj. F, oc. I.)

1. Epithelial wall of a vesicle, which at—

2, 2. Contains numerous yellow pigment granules. At the right-hand lower corner the epithelium is cut somewhat obliquely, so that its inner margin is not well defined. In the interior of the vesicle are seen numerous red blood-corpuscles.

3. Portion of the wall of an adjacent vesicle also containing many red blood-corpuscles. In this epithelial wall are likewise seen numerous pigment granules.

Fig. 14. From a transverse section of a thyroid gland of the Rook, uninjected, hardened in alcohol and stained with hæmatoxylin. Representing the epithelial walls of several vesicles containing yellow pigment-granules of various sizes.

1, 1, 1, 1, 1. Cavities of vesicles, whose contents are not represented.

2. Epithelial wall of a vesicle, seen from the surface, showing numerous pigment granules. (Obj. F, oc. I.)

Fig. 15. Large round cells from a longitudinal section of the thyroid gland of the Pigeon, injected from the artery, hardened in alcohol and stained with picrocarminate of ammonia and with hæmatoxylin. (Obj. 8, oc. I.)

Fig. 16. From a section of the thyroid gland of the Tortoise, injected with $\frac{1}{4}$ per cent. solution of nitrate of silver by the method of puncture, hardened in alcohol and stained with hæmatoxylin. Surface view of the endothelium of a large lymphatic, showing in addition to the endothelial markings areas of various size, stained darkly with nitrate of silver. (Obj. 8, oc. I.)

Fig. 17. From a longitudinal section of the thyroid gland of a Pigeon, injected by the method of puncture with $\frac{1}{4}$ per cent. solution of nitrate of silver, hardened in alcohol and stained with hæmatoxylin.

1, 1, 1. Small intervesicular blood-vessels, into which the nitrate of silver has run, containing red blood-corpuscles.

2, 2, 2. Walls of ditto stained with nitrate of silver.

3, 3, 3. Gland-vesicles stained with hæmatoxylin.

In this specimen the blood-vessels also contain coagulated fibres which are not represented in the drawing. (Obj. 6, oc. I, tube drawn out.)

Fig. 18. From a longitudinal section of the thyroid gland of the Pigeon, injected by the method of puncture with Berlin blue, hardened in alcohol and stained with hæmatoxylin.

1, 1, 1. Gland-vesicles only represented in outline.

2, 2. Two arteries seen in transverse or slightly oblique section.

3, 3. Vein containing red blood-corpuscles and partly surrounding the arteries

In addition to the blood-corpuscles the vein contains Berlin blue injection, and some homogeneous balls which are not represented in the figure. (Obj. F, oc. I.) (For explanation of these two figures see above, p. 598.)

Fig. 19. From a longitudinal section of the thyroid gland of a dog (male, aged 3 months), uninjected, hardened in alcohol. Undeveloped portion seen in section lying in a depression on the surface of the gland.

1, 1, 1. Layers of connective tissue separating the undeveloped portion (2) from the ordinary gland-tissue, and also forming septa running into the interior of the gland. The connective tissue is also seen to be continuous with the capsule of the gland. Blood-vessels are seen in it in section.

2. Section of an "undeveloped portion," with an indication of its convoluted cylinders.

3, 3, 3. Ordinary gland-tissue, the vesicles being represented diagrammatically and in outline. (Obj. O, oc. I.)

Fig. 20. From the same gland. Section of part of an "undeveloped portion" seen under a high power.

1, 1, 1. "Cylinders of cells."

2, 2, 2. Septa of connective tissue and blood-vessels separating these.

3. A "cylinder" from which the epithelial cells have partly fallen out. (Obj. F, oc. I.)

On Toroidal Functions.

XIV. *By* W. M. Hicks, *M.A., Fellow of St. John's College, Cambridge.*

Communicated by J. W. L. Glaisher, *M.A., F.R.S.*

Received February 21,—Read March 3, 1881.

The following investigation was originally undertaken as the foundation for certain researches on the theory of vortex rings, with especial reference to a theory of gravitation propounded by the author in the Proceedings of the Cambridge Philosophical Society (vol. iii., p. 276). As the results seemed interesting in themselves, and as they also serve as a basis for other investigations, more particularly in electricity and conduction of heat, I have thought it advisable to publish it as a separate paper, especially as I cannot hope to find leisure for some time to complete my original purpose.

The word "tore" is used as a name for an anchor-ring, here restricted to a circular section, and by "toroidal functions" are understood functions which satisfy Laplace's equation and which are suitable for conditions given over the surfaces of tores.

The *first* section is devoted to the general theory of the employment of two dimensional equipotential lines in certain cases as orthogonal co-ordinates in problems of three dimensions. From this we pass at once to the particular case where the two-dimensional lines are the system of circles through two fixed points and the system of circles orthogonal to them. It is shown that these satisfy the conditions of applicability. By revolution about the line through the two points we have functions suitable for problems connected with two spheres. By revolution about the line bisecting at right angles the distance between the points we have functions associated with anchor-rings or tores. By the first system it is also possible to deduce functions for what may be called a self-intersecting tore, and by the second for two intersecting spheres. A second application is made for the particular case where the opening of a tore vanishes and there is a double cuspidal point at the centre.

The *second* section is devoted to the development of zonal toroidal functions—that is, for conditions symmetrical about the axis* of a tore. It is shown that for space not containing the critical axis these are the same as zonal spherical harmonics of

* Throughout the paper the axis of a tore is taken to be the line perpendicular to its plane through its centre; the circle traced out by the centre of the generating circle of a tore will be called the circular axis, and the circle by the two points above mentioned the critical circle.

imaginary argument and (when the *whole* of space outside a tore is in question) of orders $(2n+1)/2$. For space inside a tore we have a corresponding analogy with zonal harmonics of the second kind. The properties of these functions are found to have analogies with those of the ordinary spherical harmonics, but with essential differences. The space outside a tore is different from that outside a sphere in being cyclic; in general, then, the functions for space outside will not be determinate from the surface conditions alone. The above functions are suitable only when there is no cyclic function: it is shown how to obtain a function which will complete the solution.

The *third* section deals with tesseral toroidal functions, which come into use for the most general case of non-symmetrical conditions. It is shown how the different orders and suffixes depend on each other, so that they may be calculated in terms of two. Integral expressions are also obtained, as in the second section, which are needed in finding the coefficients in expansions in series.

The *fourth* section briefly notices the functions suitable for tores without a central opening. These functions bear the same relation to the foregoing functions that cylindric harmonics (BESSEL's functions) do to spherical harmonics.

In the *fifth* section a few examples are given of the application of the method, such as the potential of a ring, the electric potential of a tore and its capacity, the electric potential of a tore and an electrified circular wire whose axis is the same as that of the tore, the potential under the influence of an electrified point arbitrarily placed, and the velocity potential for a tore moving parallel to its axis, as well as the energy of the motion.

Of previous writings on the subject, or nearly connected therewith, I am only acquainted with two. In RIEMANN's 'Gesammelte Werke'* (chap. xxiv.) is a short paper of six pages, "Ueber das Potential eines Ringes." He arrives at the same differential equation as (7) in this paper, points out that a solution can be expressed as a hypergeometric series in several ways, and that each function can be expressed in terms of two, which are elliptic integrals of the first and second kinds. The paper is a posthumous one and is not developed. There is a note on the same subject by W. D. NIVEN in the 'Messenger of Mathematics' for December, 1880. Though not bearing on the same subject, a paper may be mentioned by C. NEUMANN, "Allgemeine Lösung des Problemes über den stationären Temperaturzustand eines homogenen Körpers welcher von irgend zwei nichtconcentrischen Kugelflächen begrenzt wird" (H. W. SCHMIDT, Halle). This is a pamphlet of about 150 pages. He uses co-ordinates analogous to those in the present paper, but the method of development is very different. The functions are spherical and cylindric harmonics of real argument, and those of the second kind do not enter. He considers the stationary temperature in a shell bounded by non-concentric spheres; in an infinite medium in which are two spherical cavities; and similar cases when the boundaries touch. Its interest in con-

* The greater portion of the following pages was completed before I became acquainted with this paper of RIEMANN's or with that of NEUMANN's mentioned below.

nexion with the following pages consists chiefly in the fact that the potential is expressed in a series of the form

$$f(u.v)\Sigma F_n(u), G_n(v)$$

and that the orthogonal co-ordinates employed are closely allied.

I.

GENERAL THEORY OF CONJUGATE CURVILINEAR CO-ORDINATES IN THREE DIMENSIONS. SPECIAL CASE.

1. It is well known that if Laplace's equation be referred to a system of orthogonal co-ordinates u, v, w it takes the form

$$\frac{\delta}{\delta u}\left(\frac{U}{VW}\frac{\delta\phi}{\delta u}\right)+\frac{\delta}{\delta v}\left(\frac{V}{UW}\frac{\delta\phi}{\delta v}\right)+\frac{\delta}{\delta w}\left(\frac{W}{UV}\frac{\delta\phi}{\delta w}\right)=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where

$$U^2=\left(\frac{\delta u}{\delta x}\right)^2+\left(\frac{\delta u}{\delta y}\right)^2+\left(\frac{\delta u}{\delta z}\right)^2$$

$$V^2=\left(\frac{\delta v}{\delta x}\right)^2+\left(\frac{\delta v}{\delta y}\right)^2+\left(\frac{\delta v}{\delta z}\right)^2$$

$$W^2=\left(\frac{\delta w}{\delta x}\right)^2+\left(\frac{\delta w}{\delta y}\right)^2+\left(\frac{\delta w}{\delta z}\right)^2$$

Let us now take u, v to be any conjugate functions of ρ, z; ρ, z being the cylindrical co-ordinates of a point. Also take w to represent a series of planes through the axis of z, so that $w=\tan^{-1} y/x$.

Then u, v, w are orthogonal surfaces and

$$U^2=\left(\frac{\delta u}{\delta\rho}\right)^2+\left(\frac{\delta u}{\delta z}\right)^2=\left(\frac{\delta v}{\delta\rho}\right)^2+\left(\frac{\delta v}{\delta z}\right)^2=V^2$$

$$W^2=\frac{1}{\rho^2}$$

So that equation (1) becomes

$$\frac{\delta}{\delta u}\left(\rho\frac{\delta\phi}{\delta u}\right)+\frac{\delta}{\delta v}\left(\rho\frac{\delta\phi}{\delta v}\right)+\frac{1}{\rho\left\{\left(\frac{\delta u}{\delta\rho}\right)^2+\left(\frac{\delta u}{\delta z}\right)^2\right\}}\frac{\delta^2\phi}{\delta w^2}=0$$

In this write $\phi=\psi/\sqrt{\rho}$, then

$$\frac{\delta^2\psi}{\delta u^2}+\frac{\delta^2\psi}{\delta v^2}+\frac{\psi}{2\rho^2}\left\{\left(\frac{\delta\rho}{\delta u}\right)^2+\left(\frac{\delta\rho}{\delta v}\right)^2\right\}-\frac{\psi}{2\rho}\left(\frac{\delta^2\rho}{\delta u^2}+\frac{\delta^2\rho}{\delta v^2}\right)+\frac{1}{\rho^2\left\{\left(\frac{\delta u}{\delta\rho}\right)^2+\left(\frac{\delta u}{\delta z}\right)^2\right\}}\frac{\delta^2\psi}{\delta w^2}=0$$

Here since u, v are conjugate functions of ρ, z

$$\frac{\delta^2\rho}{\delta u^2}+\frac{\delta^2\rho}{\delta v^2}=0$$

$$\left(\frac{\delta\rho}{\delta u}\right)^2+\left(\frac{\delta\rho}{\delta v}\right)^2=\frac{1}{\left(\frac{\delta u}{\delta\rho}\right)^2+\left(\frac{\delta u}{\delta z}\right)^2}=\frac{1}{\xi^2}\text{ (say)}$$

so that

$$\frac{\delta^2\psi}{\delta u^2}+\frac{\delta^2\psi}{\delta v^2}+\frac{1}{\rho^2\xi^2}\left\{\frac{\delta^2\psi}{\delta w^2}+\tfrac{1}{4}\psi\right\}=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

By putting $\frac{\delta^2\psi}{\delta w^2}=0$ we get the equation for functions satisfying conditions symmetrical about an axis which by an obvious analogy may be called zonal functions. In general, put $\psi=\psi'\cos(mw+\alpha)$, then ψ' must satisfy

$$\frac{\delta^2\psi'}{\delta u^2}+\frac{\delta^2\psi'}{\delta v^2}-\frac{4m^2-1}{4\rho^2\xi^2}\psi'=0$$

When u, v are functions such that $1/(\rho\xi)^2$ is of the form $4(f(u)+\mathrm{F}(v))$, it is possible to obtain solutions of the form $\psi=\Sigma\mathrm{X}_{m,n}\mathrm{Y}_{m,n}\cos(mw+\alpha)$ where

$$\frac{d^2\mathrm{X}_{m,n}}{du^2}=(4m^2-1)f(u)\mathrm{X}+n^2\mathrm{X}$$

$$\frac{d^2\mathrm{Y}_{m,n}}{dv^2}=(4m^2-1)\mathrm{F}(v)\mathrm{Y}-n^2\mathrm{Y}$$

which are such that $\mathrm{X}_{m,n}$ are constant when u is constant and $\mathrm{Y}_{m,n}$ constant when v is constant.

As an instance of functions satisfying these conditions we may take the elliptic co-ordinates

$$\rho=a\cosh u\cos v \qquad z=a\sinh u\sin v$$

Here

$$\frac{1}{\rho^2\xi^2}=\frac{1}{\cos^2 v}-\frac{1}{\cosh^2 u}$$

And the equations for the functions X, Y are

$$\frac{d^2\mathrm{X}}{du^2}+\left(\frac{4m^2-1}{4\cosh^2 u}\pm n^2\right)\mathrm{X}=0$$

$$\frac{d^2\mathrm{Y}}{dv^2}-\left(\frac{4m^2-1}{4\cos^2 v}\pm n^2\right)\mathrm{Y}=0$$

The first produces functions analogous to those discussed in this paper—the second spherical harmonics of argument $\frac{\pi}{2}-v$, and order $\frac{m+1}{2}$. The surfaces $u=$ const. give confocal spheroids. Since $\sqrt{\rho}=\sqrt{a\cosh u\cos v}$, it will result that ϕ is expressed as the sum of terms of the form $\{AP(u)+BQ(u)\}\{CP'(v)+DQ'(v)\}\cos(mw+\beta)$, where P', Q' are spherical harmonics of argument $\frac{\pi}{2}-v$, and P, Q are spherical harmonics of imaginary argument.

In the applications that follow it happens that u, v are such that $\rho^2\xi^2$ is a function of u only, say $f(u)$; in this case we obtain solutions by putting $\psi'=\psi\cos(nv+\beta)$, where

$$\frac{d^2\psi}{du^2}-\left(n^2+\frac{4m^2-1}{4f(u)}\right)\psi=0 \qquad (3)$$

The solutions of this equation for $m=0$ may be called zonal functions, for $n=0$ sectorial functions, and for $m.n$ general, tesseral functions. If $U_{m.n}$, $U'_{m.n}$ are two independent solutions of this equation the general value of ϕ is given by

$$\sqrt{\rho}\phi=\Sigma\Sigma\{AU_{m.n}\cos(nv+\alpha)\cos(mw+\beta)+A'U'_{m.n}\cos(nv+\alpha')\cos(mw+\beta')\}$$

If ϕ be given over any two surfaces $u=$ const., it is clear that the constants can be determined in the above by means of FOURIER'S theorem. This will be more fully discussed in the sequel.

2. Before passing on to particular cases, there is one remarkable result to be noticed. If in the equation transformed as above, we put $\psi=\psi'\cos(\frac{1}{2}w+\gamma)$ then ψ' satisfies the equation

$$\frac{\delta^2\psi'}{\delta u^2}+\frac{\delta^2\psi'}{\delta v^2}=0$$

Hence if ψ' be any two-dimensional potential function, then $\frac{A}{\sqrt{\rho}}\psi'\cos(\frac{1}{2}w+\gamma)$ is a three-dimensional potential function. Since this expression changes sign when w increases by 2π it is not suitable for all space; but a diaphragm must be supposed to extend from the axis of z to infinity in one direction, and to be impassable. Though the result is interesting it does not seem to carry important consequences, as there is not sufficient generality in the expression. We may choose the form of the surface, and certain other conditions, but all the surface conditions are not arbitrary. Thus let us take an anchor ring divided by a plane through its axis. Let us keep the curved surface and one end at zero temperature, then the distribution of temperature at the other end is determinate though its absolute magnitude is arbitrary. To prove this, we notice that if (a, b) be the radii of the circular axis, and generating circle respectively, and r the distance of any point from the circular axis

$$\psi' = \log \frac{r}{b} = \tfrac{1}{2} \log \frac{(\rho-a)^2+z^2}{b^2}$$

and

$$t = \frac{A}{2\sqrt{\rho}} \log \frac{(\rho-a)^2+z^2}{b^2} \cos\left(\tfrac{1}{2}w+\gamma\right)$$

This is to be zero when $w=0$ $\therefore$ $\gamma=\frac{\pi}{2}$ and

$$t = \frac{A}{2\sqrt{\rho}} \log \frac{(\rho-a)^2+z^2}{b^2} \sin \frac{w}{2}$$

But now the [illegible] temperature at the other end is given by

$$t = \frac{A}{2\sqrt{\rho}} \log \frac{\overline{\rho-a}^2+z^2}{b^2}$$

leaving only the absolute magnitude A at our disposal. Further, there must be supposed a generation of heat all along the circular axis. This example serves to show the artificiality of solution given by this form.

3. For the case of an anchor ring, or tore, it is at once evident that the proper functions u, v to take are the well known ones given by

$$u+vi = \log \frac{\rho+a+zi}{\rho-a+zi}$$

viz.: v=const. a series of circles through two points $(\pm a, 0)$ and u=const. a series of circles orthogonal to them, and each containing one of the fixed points. If these be made to revolve about the line through the fixed points, we get functions proper for two spheres (u); or the surface formed by the revolution of a circle about a line cutting it (v). If they revolve about the axis of z, we get functions proper for circular tores (u); or for two intersecting spheres (v). It will be useful to set down here in a compact form, formulæ relating to these functions, which will be required later on. Most of them are easily proved and are set down without proof.

$$\left.\begin{aligned} u &= \tfrac{1}{2} \log \frac{z^2+(\rho+a)^2}{z^2+(\rho-a)^2} \\ v &= -\tan^{-1}\frac{z}{\rho+a} + \tan^{-1}\frac{z}{\rho-a} \\ &= \tan^{-1}\frac{2az}{\rho^2+z^2-a^2} \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (4)$$

$$\left.\begin{aligned} \rho+zi &= a\frac{e^{u+vi}+1}{e^{u+vi}-1} \\ \rho &= a\frac{\sinh u}{\cosh u-\cos v} \\ z &= a\frac{\sin v}{\cosh u-\cos v} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \quad (5)$$

$$\frac{du}{d\sigma}=\xi=\frac{\cosh u-\cos v}{a}=\frac{\sinh u}{\rho}$$

whence the statement made above that $\rho\xi=f(u)$.

Def. R, r be the radii of the circular axis and normal section of a tore (u); r' the radius of a sphere (v); then

$$\left.\begin{aligned} &R^2-r^2=a^2 \\ &\cosh u=\frac{R}{r} \\ &\sinh u=\frac{\sqrt{R^2-r^2}}{r}=\frac{a}{r} \\ &\sin v=\frac{a}{r'} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \quad (6)$$

Further, if r, the radius of a tore to a point P (u, v) make an angle θ with the plane of the ring

$$\cos v=-\frac{r-R\cos\theta}{R-r\cos\theta}$$

$$\sin v=\frac{\sqrt{R^2-r^2}\sin\theta}{R-r\cos\theta}$$

With the above values of (u, v) the general equation for toroidal tesseral functions is

$$\frac{d^2\psi}{du^2}-n^2\psi+\frac{4m^2-1}{4\sinh^2 u}\psi=0 \quad \ldots\ldots\ldots\ldots \quad (7)$$

There is one case in which the functions used above become nugatory—viz.: when a is zero, or the tores are such that $R=r$ and they touch themselves at the origin. In this case the proper curves are the two orthogonal families of circles, touching, the one set the axis of z, and the other the axis of ρ at the origin—viz.:

$$\left.\begin{aligned}
&(u+vi)(\rho+zi)=a\\
&u=\frac{a\rho}{\rho^2+z^2}\\
&v=-\frac{az}{\rho^2+z^2}\\
&\rho=\frac{au}{u^2+v^2}\\
&z=-\frac{av}{u^2+v^2}\\
&\frac{du}{d\eta}=\xi=\frac{a}{\rho^2+z^2}=\frac{u^2+v^2}{a}\\
&\qquad=\frac{u}{\rho}
\end{aligned}\right\}\quad\ldots\ldots\ldots\quad(8)$$

and

$$\frac{d^2\psi}{du^2}-n^2\psi-\frac{4m^2-1}{4u^2}\psi=0$$

It will be shown that between the latter surfaces and tores there is a similar relation to that between cylinders and spheres, and between the functions to that between Spherical Harmonics and Bessel's functions.

4. The potential due to a ring of radius b, centre at $(o.z')$ and plane perpendicular to axis of z, is

$$\phi=\int_0^\pi\frac{bd\theta}{\sqrt{(z-z')^2+\rho^2+b^2-2b\rho\cos\theta}}$$

$$=\sqrt{\frac{b}{2\rho}}\int_0^\pi\frac{d\theta}{\sqrt{\alpha-\cos\theta}}$$

In the case where it is the critical circle

$$\alpha=\frac{z^2+\rho^2+a^2}{2a\rho}=\coth u$$

and here

$$\psi=\int_0^\pi\frac{d\theta}{\sqrt{\coth u-\cos\theta}}$$

In general the distance between two points is $(z-z')^2+\rho^2+\rho'^2-2\rho\rho'\cos(w-w')$, which expressed in bipolar co-ordinates becomes

$$\frac{2a^2}{(\cosh u-\cos v)(\cosh u'-\cos v')}\{\cosh u\cosh u'-\cos(v-v')-\sinh u\sinh u'\cos(w-w')\}$$

II.

ZONAL TOROIDAL FUNCTIONS.

5. In the case where the conditions are symmetrical about the axis, ϕ is independent of w and is of the form

$$\phi=\sqrt{\frac{\cosh u-\cos v}{\sinh u}}\,\Sigma\psi_n\cos(nv+\alpha)$$

where ψ_n is the general integral involving two arbitrary constants of the equation

$$\frac{d^2\psi}{du^2}-n^2\psi+\frac{\psi}{4\sinh^2 u}=0$$

From the potential of a ring, at the end of the last section, it is at once seen that a particular integral for space, not including the critical circle, when $n=0$ is

$$\psi_0=\sqrt{\sinh u}\int_0^\pi\frac{d\theta}{\sqrt{\cosh u-\sinh u\cos\theta}}$$

In the same way, calling the potential of the ring ϕ, it may be shown by finding $\frac{\partial\phi}{\partial z}$ that

$$\psi_1=\sqrt{\sinh u}\int_0^\pi\frac{d\theta}{(\cosh u-\sinh u\cos\theta)^{\frac{3}{2}}}$$

From analogy with this we might assume

$$\psi_n=\sqrt{\sinh u}\int_0^\pi\frac{d\theta}{(\cosh u-\sinh u\cos\theta)^p}$$

and by substituting we should find it possible by putting $p=\frac{2n+1}{2}$ or $-\frac{2n-1}{2}$ to satisfy the equation. But the following, by making use of theorems already proved for zonal harmonics, seems to be more direct. Putting, in the differential equation,

$$\psi=\sqrt{\sinh u}.\mathrm{P}$$

$$\frac{d^2\mathrm{P}}{du^2}+\coth u\frac{d\mathrm{P}}{du}-(n-\tfrac{1}{2})(n+\tfrac{1}{2})\mathrm{P}=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

whence it is at once evident that P_n is a zonal spherical harmonic of degree $\frac{2n+1}{2}$, with a pure imaginary for argument. HEINE, in his 'Handbuch der Kugelfunctionen,'

has to some extent considered spherical harmonics with imaginary argument, but he has not developed them, at least for fractional indices, in a form suitable for application here. Consequently they will be here considered independently and with especial reference to physical applications. Hereafter, C, S will be used in general to represent cosh u, sinh u, respectively.

We have then in general

$$\phi=\sqrt{C-\cos v}\,\Sigma(A_n P_n+B_n Q_n)\cos(nv+\alpha)$$

where P_n, Q_n are two independent integrals of equation (9). We first discuss the integral already obtained

$$P_n=\int_0^\pi \frac{d\theta}{(C-S\cos\theta)^{\frac{2n+1}{2}}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

It is well known that this integral is the same as

$$\int_0^\pi (C-S\cos\theta)^{\frac{2n-1}{2}}d\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

the second solution obtained above; this may be easily shown by the transformation $(C-S\cos\theta)(C-S\cos\theta')=1$ or by means of the sequence equation (14) below.

6. *Discussion of* P_n.

We have

$$\frac{dP_n}{du}=-\left(\frac{2n+1}{2}\right)\int_0^\pi \frac{S-C\cos\theta}{(C-S\cos\theta)^{\frac{2n+3}{2}}}$$

Whence

$$\frac{2S}{2n+1}\frac{dP_n}{du}=P_{n+1}-CP_n \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

Similarly from

$$\frac{dP_n}{du}=\frac{2n-1}{2}\int_0^\pi (C-S\cos\theta)^{\frac{2n-3}{2}}(S-C\cos\theta)\delta\theta$$

$$\frac{2S}{2n-1}\frac{dP_n}{du}=CP_n-P_{n-1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

Combining (12) and (13) we have

$$(2n+1)P_{n+1}-4nCP_n+(2n-1)P_{n-1}=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

This sequence equation may also be deduced at once from (10) or (11).

In (14) put

$$P_n=\frac{(2n-2)(2n-4)\ldots 2}{(2n-1)(2n-3)\ldots 3\,.\,1}\,u_n$$

with

$$P_0=u_0,\quad P_1=u_1$$

then

$$u_{n+2}-2Cu_{n+1}+\frac{(2n+1)^2}{2n(2n+2)}u_n=0$$

or

$$u_{n+2}=2Cu_{n+1}-c_nu_n$$

where

$$c_n=\frac{(2n+1)^2}{2n(2n+2)}=\frac{(2n+1)^2}{(2n+1)^2-1}=1+\tfrac{1}{2}\left(\frac{1}{2n}-\frac{1}{2n+2}\right)$$

and

$$c_0=\tfrac{1}{2}$$

It is clear from this that u_n is of the form $\alpha_nu_1-\beta_nu_0$ where α_n,β_n are rational integral algebraical functions of $2C$; α_n of degree $n-1$, and β_n of degree $n-2$. The first three values are (writing $2C=x$)

$$u_0=u_0$$
$$u_1=u_1$$
$$u_2=xu_1-\tfrac{1}{2}u_0$$
$$u_3=(x^2-c_1)u_1-\tfrac{1}{2}xu_0$$

We can now show that α_n, β_n are of the form

$$\alpha_n=a_{n,0}x^{n-1}+a_{n,1}x^{n-3}+a_{n,2}x^{n-5}+\ldots+a_{n,r}x^{n-2r-1}+\ldots$$

For supposing α_n of this form, *i.e.*, wanting every other power of x, it follows at once that α_{n+1} is of the same form, and it is seen above that α_3 is of this form, whence the statement is generally true, and so also for β_n.

Now α_n satisfies the equation

$$\alpha_n=x\alpha_{n-1}-c_{n-2}\alpha_{n-2}\quad\ldots\ldots\ldots\quad(15)$$

with

$$\alpha_0=0\quad\alpha_1=1$$

Hence substituting the above value for α_n we must have

$$a_{n,r}=a_{n-1,r}-c_{n-2}a_{n-2,r-1}\quad\ldots\ldots\ldots\quad(16)$$

also

$$a_{n,0}=a_{n-1,0}=\ldots=1$$

Hence

$$a_{n,r}=-(c_{n-2}\,a_{n-2,r-1}+c_{n-3}\,a_{n-3,r-1}+\ldots+c_{2r-1}a_{2r-1,r-1})$$

From this

$$a_{n,1}=-(c_{n-3}+c_{n-3}+\ldots+c_1)$$

$$a_{n,2}=\{c_{n-2}(c_{n-4}+\ldots+c_1)+c_{n-3}(c_{n-5}+\ldots+c_1)+\ldots+c_3c_1$$

= Sum of products two and two together, with the exception of all products where the subscript numbers are successive.

Assume that $(-)^r a_{n,r}$=sum of products of the c_m up to c_{n-2}, r together with the exception of any in which successive subscripts occur. Then

$$\begin{aligned}a_{n,r+1}=(-)^{r+1}\{&c_{n-2}(\text{prod. up to } c_{n-4},\ r \text{ together, \&c. }\ldots)\\&+c_{n-3}(\quad,,\quad,,\quad c_{n-5}\ldots\quad)\\&+\ldots\quad\}\\=(-)^{r+1}\{&\text{Prod. } (r+1) \text{ together up to } c_{n-2} \text{ without successive subscripts.}\}\end{aligned}$$

Whence by induction the assumption is seen to be universally true. It may be thus stated, $a_{n,r}$ is the sum r together of the terms

$$\frac{3^2}{2.4},\ \frac{5^2}{4.6},\ \frac{7^2}{6.8},\ \ldots\ \frac{(2n-3)^2}{(2n-2)(2n-4)},$$

all products being thrown out in which, regarding the numbers in the denominators as undecomposable, a square occurs in the denominator.

We have

$$a_{n,0}=\ 1$$

$$a_{n,1}=-\frac{(4n-3)(n-2)}{4(n-1)}$$

This result is of very little use for application. If the co-efficients (α_n) are needed for particular values of x they can be very rapidly calculated by means of equation (15), while if their general values are to be tabulated, equation (16) will serve to calculate them in succession.

Further, $2\beta_n$ is the same kind of function as α_n, in every way except that it does not contain c_1; in fact $2\beta_n$ is the same function of $c_3, c_5, \ldots c_{n-2}$, that α_{n-1} is of $c_1, c_3, \ldots c_{n-3}$; calling this α'_{n-1} we can then write

$$u_n=\alpha_n u_1-\tfrac{1}{2}\alpha'_{n-1}u_0$$

u_1, u_0 may be expressed as elliptic integrals, viz.:

$$\left.\begin{aligned} u_0&=\int_0^\pi \frac{d\theta}{\sqrt{C-S\cos\theta}}=2\sqrt{k}\int_0^{\frac{\pi}{2}}\frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}=2\sqrt{k}\,F\\ u_1&=\int_0^\pi \sqrt{C-S\cos\theta}\,d\theta=\frac{2}{\sqrt{k}}\int_0^{\frac{\pi}{2}}\sqrt{1-k^2\sin^2\theta}\,d\theta=\frac{2}{\sqrt{k}}E\end{aligned}\right\} \quad . \quad . \quad . \quad (17)$$

where

$$k^2=\frac{2S}{C+S} \qquad k'^2=\frac{1}{(C+S)^2}$$

or

$$k^2=1-e^{-2u} \qquad k'^2=e^{-2u}$$

and

$$x=2C=k'+\frac{1}{k'}$$

Hence

$$P_n=2\frac{(2n-2)(2n-4)\ldots 2}{(2n-1)\ldots 3}\left(\frac{\alpha_n}{\sqrt{k'}}E-\tfrac{1}{2}\sqrt{k'}.\alpha'_{n-1}F\right) \quad . \quad . \quad . \quad . \quad (18)$$

where we may suppose the numerical factor dropped if we are dealing with the differential equation, but not if we are dealing with the sequence equation.

The value of P_n when $u=0$ is π

" " $u=\infty$ is ∞

These statements are at once seen to be true. Since u becomes infinite along the critical circle it follows that the P_n are not the suitable functions to use by which to express functions which are finite in spaces containing the critical circle, *i.e.*, within any tore. But it is finite and continuous for all space outside any tore.

7. If we put for P_n in equation (9) $P_nQ'_n$ we find in the usual way

$$P_nQ'_n=BP_n+AP_n\int_0^u\frac{du}{S^2P_n^2}$$

Regarded as an analytical solution of the equation this is complete, but in this form it is altogether useless for application. Now HEINE* has shown that the spherical harmonic of the second kind is expressible in the form

$$Q_n(x)=\int_0^\infty\frac{d\theta}{(x-\sqrt{x^2-1}\cosh\theta)^n}$$

* 'Kugelfunctionen,' Kap. iii.

We first show that this with a modification satisfies the equation (9). For putting

$$Q_n=\int_0^\infty \frac{d\theta}{(C+S\cosh\theta)^{\frac{2n+1}{2}}} \quad \ldots\ldots\ldots\ldots \quad (19)$$

$$\frac{dQ_n}{du}=-\frac{2n+1}{2}\int_0^\infty \frac{S+C\cosh\theta}{(C+S\cosh\theta)^{\frac{2n+3}{2}}}d\theta$$

$$\frac{d^2Q_n}{du^2}=\left(\frac{2n+1}{2}\right)^2 Q_n-\frac{2n+1}{2S}\int_0^\infty \sinh\theta\frac{d}{d\theta}(C+S\cosh\theta)^{-\frac{2n+3}{2}}d\theta$$

$$=\left(\frac{2n+1}{2}\right)^2 Q_n+\frac{2n+1}{2S}\int_0^\infty \frac{\cosh\theta d\theta}{(C+S\cosh\theta)^{\frac{2n+3}{2}}}$$

$$\frac{d^2Q_n}{du^2}+\coth u\frac{dQ}{du}=\left(\frac{2n+1}{2}\right)^2 Q_n-\frac{2n+1}{2S}\int_0^\infty \frac{CS+(C^2-1)\cosh\theta}{(C+S\cosh\theta)^{\frac{2n+3}{2}}}d\theta$$

$$=\frac{4n^2-1}{4}Q_n$$

Here also as in the case of the P functions it can be easily shown that

$$\frac{2S}{2n+1}\frac{dQ_n}{du}=Q_{n+1}-CQ_n$$

$$\frac{2S}{2n-1}\frac{dQ_n}{du}=CQ_n-Q_{n-1}$$

and

$$(2n+1)Q_{n+1}-4nCQ_n+(2n-1)Q_{n-1}=0$$

Hence as before

$$Q_n=\frac{(2n-2)\ldots 2}{(2n-1)\ldots 3}(\alpha_n v_1-\tfrac{1}{2}\alpha'_{n-1}v_0)$$

where

$$v_0=\int_0^\infty \frac{d\theta}{\sqrt{C+S\cosh\theta}}$$

$$v_1=\int_0^\infty \frac{d\theta}{(C+S\cosh\theta)^{\frac{1}{2}}}$$

In these change θ into 2θ, then

$$v_0=2\int_0^\infty \frac{d\theta}{\sqrt{\mathrm{C}-\mathrm{S}+2\mathrm{S}\cosh^2\theta}}$$

$$v_1=2\int_0^\infty \frac{d\theta}{(\mathrm{C}-\mathrm{S}+2\mathrm{S}\cosh^2\theta)^{\frac{1}{2}}}$$

Again, write $\cosh\theta=\sec\phi$, then $\sinh\theta=\tan\phi$, $d\theta=\sec\phi d\phi$, and when $\theta=0$ or ∞, $\phi=0$ or $\frac{\pi}{2}$

Hence

$$v_0=2\int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{\mathrm{C}+\mathrm{S}-(\mathrm{C}-\mathrm{S})\sin^2\phi}}$$

$$=2\sqrt{k}\mathrm{F}' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

Also

$$v_1=2\int_0^{\frac{\pi}{2}} \frac{\cos^2\phi d\phi}{\{\mathrm{C}+\mathrm{S}-(\mathrm{C}-\mathrm{S})\sin^2\phi\}^{\frac{1}{2}}}$$

$$=\frac{2}{\sqrt{k}}\int_0^{\frac{\pi}{2}} \frac{k'^2-k'^2\sin^2\phi}{(1-k'^2\sin^2\phi)^{\frac{1}{2}}}d\phi$$

$$=\frac{2\mathrm{F}'}{\sqrt{k}}-\frac{2k^2}{\sqrt{k}}\int_0^{\frac{\pi}{2}} \frac{d\phi}{(1-k'^2\sin^2\phi)^{\frac{1}{2}}}$$

Now

$$k'\frac{d\mathrm{F}'}{dk'}=\int_0^{\frac{\pi}{2}} \frac{k'^2\sin^2\phi d\phi}{(1-k'^2\sin^2\phi)^{\frac{1}{2}}}$$

$$=\int_0^{\frac{\pi}{2}} \frac{d\phi}{(1-k'^2\sin^2\phi)^{\frac{1}{2}}}-\mathrm{F}'$$

and

$$k'\frac{d\mathrm{F}'}{dk'}=\frac{1}{k^2}\mathrm{E}'-\mathrm{F}'$$

$$\therefore v_1=\frac{2}{\sqrt{k}}(\mathrm{F}'-\mathrm{E}') \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

and finally

$$\mathrm{Q}_n=2\frac{(2n-2)\ldots 2}{(2n-1)\ldots 3}\left\{\frac{\alpha_n}{\sqrt{k}}(\mathrm{F}'-\mathrm{E}')-\tfrac{1}{2}\alpha'_{n-1}\sqrt{k}\mathrm{F}'\right\} \quad . \quad . \quad . \quad . \quad (22)$$

The value of Q_n for $u=0$ is ∞, and for $u=\infty$ is zero. Hence Q_n is suitable for space within a tore, and not for space including the axis.

8. The foregoing value of Q_n has been obtained from analogy with that for P_n; but

in the same way as P_n (for space outside a tore) was obtained from the potential of a ring, so also may Q_n be determined from the potential of a point at the origin, for space not containing it, *i.e.*, for space within a tore. For the inverse distance of a point from the origin is

$$\frac{1}{a}\sqrt{\frac{C-\cos v}{C+\cos v}}$$

Hence

$$\frac{1}{\sqrt{C+\cos v}}=\Sigma(A_nP_n+B_nQ_n)\cos(nv+\alpha)$$

Now, firstly, since this is to be finite throughout all space not including the axis $A_n=0$. Also it is clear since $C>\cos v$ that $1/\sqrt{C+\cos v}$ can be expanded in a series of powers of $\cos v$, and therefore in a series of cosines of multiple angles only. If this be done the coefficient of $\cos nv$ must be B_nQ_n.

Hence by FOURIER's theorem

$$\frac{\pi}{2}B_nQ_n=\int_0^\pi\frac{\cos n\theta d\theta}{\sqrt{C+\cos\theta}}$$

If we define Q_n so as to make $\pi B_n/2=(-1)^n\sqrt{2}$, then

$$\sqrt{2}Q_n=(-)^n\int_0^\pi\frac{\cos n\theta d\theta}{\sqrt{C+\cos\theta}}$$

$$=\int_0^\pi\frac{\cos n\theta}{\sqrt{C-\cos\theta}}d\theta. \quad \ldots\ldots\ldots\ldots \quad (23)$$

We will now show that this expression for Q_n agrees with the former one. Integrating by parts and dropping $\sqrt{2}$ as unnecessary in the sequence equation

$$2nQ_n(-)^n=-\int_0^\pi\frac{\sin n\theta\sin\theta}{(C+\cos\theta)^{\frac{1}{2}}}d\theta$$

also

$$(-)^nQ_n=\int_0^\pi\frac{\cos n\theta(C+\cos\theta)}{(C+\cos\theta)^{\frac{1}{2}}}d\theta$$

$$=C\int_0^\pi\frac{\cos n\theta}{(C+\cos\theta)^{\frac{1}{2}}}+\int_0^\pi\frac{\cos n\theta\cos\theta}{(C+\cos\theta)^{\frac{1}{2}}}$$

Hence

$$(2n+1)Q_n(-)^n=\quad C\int_0^\pi\frac{\cos n\theta}{(C+\cos\theta)^{\frac{1}{2}}}+\int_0^\pi\frac{\cos(n+1)\theta}{(C+\cos\theta)^{\frac{1}{2}}}$$

$$(2n-1)Q_n(-)^n=-C\int_0^\pi\frac{\cos n\theta}{(C+\cos\theta)^{\frac{1}{2}}}-\int_0^\pi\frac{\cos n-1\theta}{(C+\cos\theta)^{\frac{1}{2}}}$$

$$\therefore (2n-1)Q_{n-1}+(2n+1)Q_{n+1}= (-)^{n+1}C\int_0^\pi \frac{\cos n-1\theta-\cos n+1\theta}{(C+\cos\theta)^{\frac{1}{2}}}$$

$$=2(-)^{n+1}C\int_0^\pi \frac{\sin n\theta \sin\theta}{(C+\cos\theta)^{\frac{1}{2}}}d\theta$$

$$=4nCQ_n.$$

The same sequence equation as before. Hence it is only necessary further to show that Q_0, Q_1 are the same in the two cases.

Now

$$Q_0\sqrt{2}=\int_0^\pi \frac{d\theta}{\sqrt{C+\cos\theta}}$$

$$=\int_0^\pi \frac{d\theta}{\sqrt{C+1-2\sin^2\frac{\theta}{2}}}$$

$$=\frac{2}{\sqrt{C+1}}\int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-\lambda'^2\sin^2\theta}}$$

where

$$\lambda'^2=\frac{2}{C+1}$$

If C, S be eliminated between λ' and $k^2=2S/(C+S)$ there result the equations

$$k=\frac{2\sqrt{\lambda}}{1+\lambda}, \quad k'=\frac{1-\lambda}{1+\lambda}$$

Hence by the second quadric transformation

$$F_{\lambda'}=(1+k')F'$$

and since

$$\lambda'=2\sqrt{k}/(1+k')$$

$$Q_0=2\sqrt{k'}F'.$$

Again

$$-Q_1\sqrt{2}=\lambda'\sqrt{2}\int_0^{\frac{\pi}{2}} \frac{\cos 2\theta d\theta}{\sqrt{1-\lambda'^2\sin^2\theta}}$$

$$=\frac{\sqrt{2}}{\lambda'}\{2E_{\lambda'}-(1+\lambda^2)F_{\lambda'}\}$$

Now

$$E(\lambda')=\lambda'\lambda^2\frac{dF(\lambda')}{d\lambda'}+\lambda^2F(\lambda')$$

$$=\lambda^2\left\{\lambda'\frac{dk'}{d\lambda'}\frac{d}{dk'}(1+k')F'+(1+k')F'\right\}$$

which on reduction and substitution for $\frac{dF'}{dk'}$ becomes

$$E(\lambda')=\frac{1}{1+k'}(2E'-k^2F')$$

$$\therefore -Q_1\sqrt{2}=\frac{1}{\sqrt{2k'}}\{4E'-2k^2F'-2(1+k'^2)F'\}$$

$$=\frac{2\sqrt{2}}{\sqrt{k'}}(E'-F')$$

and

$$Q_n=2\frac{(2n-2)\dots 2}{(2n-1)\dots 3}\left\{\frac{\alpha_n}{\sqrt{k'}}(F'-E')-\tfrac{1}{2}\alpha'_{n-1}\sqrt{k'}F'\right\}$$

We have in fact proved that

$$\int_0^\infty \frac{d\theta}{(\cosh u+\sinh u.\cosh\theta)^{\frac{2n+1}{2}}}=(-1)^n\sqrt{2}\int_0^\pi \frac{\cos n\theta d\theta}{\sqrt{\cosh u+\cos\theta}}$$

By means of the identity

$$EF'+E'F-FF'=\frac{\pi}{2}$$

or

$$E'-F'=\frac{1}{F}\left(\frac{\pi}{2}-EF'\right)$$

Q_n can be expressed in the following manner, viz.:

$$Q_n=2\frac{(2n-2)\dots 2}{(2n-1)\dots 3}\left\{\frac{\pi\alpha_n}{2F\sqrt{k'}}-\frac{F'}{F}\left(\frac{\alpha_n E}{\sqrt{k'}}+\tfrac{1}{2}\alpha'_{n-1}F\right)\right\}$$

9. The following relations between P and Q functions will be useful in applications, viz.:

$$\left.\begin{aligned}
&(\alpha)\quad P_{n+1}Q_n-P_nQ_{n+1}=\frac{2\pi}{2n+1}\\
&(\beta)\quad P'_nQ_n-P_nQ'_n=\frac{\pi}{S}\\
&(\gamma)\quad P'_nQ'_{n+1}-P'_{n+1}Q'_n=(2n+1)\frac{\pi}{2}
\end{aligned}\right\}\quad\dots\dots\quad(24)$$

They are easily proved, for substituting for P_{n+1}, Q_{n+1} from their sequence equations it follows that

$$\begin{aligned}(2n+1)(P_{n+1}Q_n-P_nQ_{n+1})&=(2n-1)(P_nQ_{n-1}-P_{n-1}Q_n)\\&=P_1Q_0-P_0Q_1\\&=4\{EF'-F(F'-E')\}\\&=4(EF'+FE'-FF')\\&=2\pi\end{aligned}$$

Again

$$\begin{aligned}2S(P'_nQ_n-P_nQ'_n)&=(2n+1)\{Q_n(P_{n+1}-CP_n)-P_n(Q_{n+1}-CQ_n)\}\\&=(2n+1)(P_{n+1}Q_n-P_nQ_{n+1})\\&=\pi\end{aligned}$$

In a similar way (γ) may also be proved.

10. As bearing on the question of the convergency or divergency of series occurring in any investigation it will be important to consider the values of P_nQ_n when n is infinite. Taking the expression for P_n

$$P_n=\int_0^\pi (C-S\cos\theta)^{\frac{2n-1}{2}}d\theta$$

it is clear at once that

$$P_{n+1}<(C+S)P_n>P_n$$

Further since P_n increases with u, $\frac{dP}{du}$ is positive, hence $P_{n+1}>CP_n$

Also from

$$Q_n=\int_0^\infty \frac{d\theta}{(C+S\cosh\theta)^{\frac{2n+1}{2}}}$$

$$Q_{n+1}<\frac{1}{C+S}Q_n$$

Also since Q_n decreases with u, $\frac{dQ}{du}$ is negative, and therefore $Q_{n-1}>CQ_n$

Hence

$$P_{n+1}Q_{n+1}<P_nQ_n$$

but tends to the limit unity, so that the series

$$\Sigma P_nQ_n \text{ is divergent.}$$

But the series

$$\Sigma P_nQ_n\cos n(v+\alpha) \text{ is convergent,}$$

except when $v+\alpha=0$.

Further if $u' > u$ (P', Q', here standing for P(u'), Q(u'))

$$P_{n+1}Q'_{n+1} < \frac{C+S}{C'+S'} P_n Q'_n$$

Hence the series

$$\Sigma P_n Q'_n < \Sigma \left(\frac{C+S}{C'+S'} \right)^n$$

and is therefore convergent. Much more then is the series $\Sigma P_n Q'_n \cos u(v+\alpha)$ convergent.

Again if $u' < u$

$$P'_{n+1}Q_{n+1} < \frac{C'+S'}{C+S} P'_n Q_n$$

and as before, the series $\Sigma P'_n Q_n \cos n(v+\alpha)$ is always convergent.

11. Both the functions $P_n Q_n$, except along the critical circle and axis respectively, make ϕ finite, continuous, and single valued when n is integral. The first statement has already been proved, the second follows from the way in which $\frac{dP}{du}, \frac{dQ}{du}$ are expressible in terms of two successive P_n or Q_n, and the third is seen to be at once true by integrating $\frac{\delta\phi}{\delta v}$ round a circuit lying on any tore $u =$ constant, when $\int \frac{d\phi}{d\theta} d\theta$ is seen to vanish. Now the space is a cyclic one. Hence the above functions are not suitable for expressing any general conditions in the space without a tore, though they are suitable for any given surface conditions whatever.

Still keeping to physical analogies in order to obtain solutions suitable to this case, we will consider the potential due to a vortex ring or electric current along the critical circle. This would give cyclic functions, but also certain surface conditions. In any particular case then it will be necessary to take account of these surface conditions by means of the P_n or Q_n. This potential is measured by the solid angle subtended by the ring.

The (solid angle) $\times \mu$ can be expressed in the form ;—*c.s.* denoting cos v, sin v,—

$$2\mu\pi - \sqrt{2}\mu \sin v \sqrt{C-c} \int_0^\pi \frac{C+c-S\cos\phi}{s^2+S^2\sin^2\phi} \frac{d\phi}{\sqrt{C-S\cos\phi}}$$

or

$$2\mu\pi - \mu \frac{k \sin v}{\sqrt{S}} \left\{ \frac{\sqrt{C+c}+\sqrt{C-c}}{\sqrt{C^2-c^2}-S} \Pi(n_1.k) \right.$$

$$\left. + \frac{\sqrt{C+c}-\sqrt{C-c}}{\sqrt{C^2-c^2}+S} \Pi(n_2.k) \right.$$

where

$$n_1 = \frac{2S}{\sqrt{C^2-c^2}-S} \qquad n_2 = -\frac{2S}{\sqrt{C^2-c^2}+S}$$

To complete the general expression for ψ we must therefore add a term

$$A \sin v \int_0^{\pi} \frac{\cosh u + \cos v - \sinh u \cos\phi}{\sin^2 v + \sinh^2 u \,.\, \sin^2\phi} \frac{d\phi}{\sqrt{\cosh u - \sinh u \cos\phi}} \quad . \quad . \quad . \quad . \quad (25)$$

We shall denote this by the letter $A\Omega$, so that the solid angle varies as $\Omega\sqrt{C-c}$.

III.

SECTORIAL AND TESSERAL FUNCTIONS.

12. The differential equation which has to be considered in the general case is

$$\frac{d^2\psi}{du^2} - n^2\psi - \frac{4m^2-1}{4\sinh^2 u}\psi = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

which in the case of sectorial functions becomes

$$\frac{d^2\psi}{du^2} - \frac{4m^2-1}{4\sinh^2 u}\psi = 0$$

In the rest of this paper we shall call n the order of the function and m the rank.

Calling the solution of (7) $\psi_{m,n}$ we proceed to show how $\psi_{m,n}$ can be expressed in terms of $\psi_{m,0}$, $\psi_{m,1}$.

Dropping the m for the time, assume

$$\psi_{n+1} = \psi_n \,.\, f(u) + \frac{d\psi_n}{du}\phi(u)$$

Then writing $(4m^2-1)/4 = \lambda$, and substituting in the equation which ψ_{n+1} satisfies, making use of the equation for ψ_n to express $\frac{d^2\psi_n}{du^2}$, and $\frac{d^3\psi_n}{du^3}$ in terms of ψ_n and $\frac{d\psi_n}{du}$ we shall get

$$\psi''_{n+1} - (n+1)^2\psi_{n+1} + \frac{\lambda}{S^2}\psi_{n+1}$$
$$= \psi_n\left\{-(2n+1)f + f''' + 2n^2\phi' - \frac{2\lambda}{S}\frac{d}{du}\left(\frac{\phi}{S}\right)\right\} + \psi_n'\{2f'' + \phi'' - \overline{2n+1}|\phi\}$$

Now choose f, ϕ, so that

$$\left.\begin{aligned} f''' - (2n+1)f + 2n^2\phi' - \frac{2\lambda}{S}\frac{d}{du}\left(\frac{\phi}{S}\right) &= 0 \\ \phi'' - (2n+1)\phi + 2f' &= 0 \end{aligned}\right\}$$

If we try $\phi=AS$, we shall find that both the equations

$$\left.\begin{aligned} f''-(2n+1)f+2n^2AC&=0\\ 2f'-2nAS&=0\end{aligned}\right\}$$

can be satisfied simultaneously if $f=nAC$.

Hence, whatever λ be, the equation

$$\psi_{n+1}=A\left(nC\psi_n+S\frac{d\psi_n}{du}\right)$$

holds.

Again, we may also determine f, ϕ, so that

$$\psi_{n-1}=\psi_n f+\frac{d\psi_n}{du}\phi$$

In this case the equations for f and ϕ are

$$\left.\begin{aligned} f''+(2n-1)f+2n^2\phi'-\frac{2\lambda}{S}\frac{d}{du}\left(\frac{\phi}{S}\right)&=0\\ \phi''-(2n-1)\phi+2f''&=0\end{aligned}\right\}$$

which are satisfied by $\phi=BS$, $f=-nBC$.

Hence, $\psi_{n-1}=B(-nC\psi_n+S\psi'_n)$.

The toroidal functions themselves are $\psi/\sqrt{S}$, and the two particular integrals are represented by $P_{m,n}$, $Q_{m,n}$. For these functions the above equations become

$$\begin{aligned} P_{m,n+1}&=A_n\{2SP'_{m,n}+(2n+1)CP_{m,n}\}\\ P_{m,n-1}&=B_n\{2SP'_{m,n}-(2n-1)CP_{m,n}\}\end{aligned}$$

and similar equations for the Q.

Since the solutions $P_{m,n}$ of the differential equation are multiplied by an arbitrary constant, we may, when we confine ourselves to one of the above equations, put A or B=1, and after solving the equation of mixed differences multiply the result by an arbitrary constant. But if we wish to combine both formulæ so as to eliminate the differential co-efficient in them, then the P in both must be the same, and a relation will hold between A and B. This we proceed to find. Dropping the (m) as unnecessary, write

$$P_n=A_{n-1}\{2SP'_{n-1}+(2n-1)CP_{n-1}\}$$

and substitute therein

$$P_{n-1}=B_n\{2SP'_n-(2n-1)CP_n\}$$

Whence

$$P_n = A_{n-1}B_n 4S^2\left(P''_n + \frac{C}{S}P'_n - \frac{4n^2-1}{4}P_n - \frac{(2n-1)^2}{4S^2}\right)$$

which since

$$P''_n + \frac{C}{S}P'_n - \frac{4n^2-1}{4}P_n - \frac{m^2}{S^2}P_n = 0$$

becomes

$$P_n = \{4m^2 - (2n-1)^2\}A_{n-1}B_nP_n$$

Hence

$$A_{n-1}B_n = \frac{1}{(2m+n-1)(2m-n+1)}$$

If we choose

$$\frac{1}{A_{n-1}} = 2(m+n)-1$$

then

$$\frac{1}{B_n} = 2(m-n)-1$$

these conditions are satisfied, and the formulæ agree with those found for the zonal function when $m=0$. Hence

$$\left.\begin{aligned} 2SP'_{m,n} &= (2m+2n+1)P_{m,n+1} - (2n+1)CP_{m,n} \\ 2SP'_{m,n} &= (2m-2n+1)P_{m,n-1} + (2n-1)CP_{m,n} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (26)$$

From this there follows at once the sequence equation

$$(2m+2n+1)P_{m,n+1} - 4nCP_{m,n} + (2n-1-2m)P_{m,n-1} = 0 \quad . \quad . \quad . \quad (27)$$

In this write

$$P_{m,n} = \frac{2^{n-1}\lfloor n-1}{(2m+2n-1)(2m+2n-3)\ldots(2m+1)}u_{m,n}$$

Then

$$u_{m,n+1} - 2Cu_{m,n} + \frac{(2n-1)^2-4m^2}{2n(2n-2)}u_{m,n-1} = 0$$

whence, if

$$c_{m,n} = \frac{(2n-1)^2-4m^2}{(2n-1)^2-1}$$

and $\alpha_{m,n}$, $\alpha'_{m,n-1}$ are the same functions of $c_{m,n}$, &c., as α_n, α'_{n-1} are of c_n

$$P_{m,n} = \frac{(2n-2)(2n-4)\ldots 2}{(2m+2n-1)(2m+2n-3)\ldots(2m+1)}\{\alpha_{m,n}P_{m,1} - \tfrac{1}{2}\alpha'_{m,n-1}P_{m,0}\} \quad (28)$$

These formulæ hold for the two particular integrals $P_{m,n}$ and $Q_{m,n}$, and they express the tesseral function of any order and rank in terms of sectorial functions

and tesseral functions of the first order and same rank. In the same way as was proved in the case of zonal functions, it may be shown that

$$P_{m,n+1}Q_{m,n}-P_{m,n}Q_{m,n+1}$$
$$=\frac{(2n-1-2m)(2n-3-2m)\ldots(1-2m)}{(2n+1+2m)(2n-1+2m)\ldots(3+2m)}(P_{m,1}Q_{m,0}-P_{m,0}Q_{m,1}) \quad . \quad . \quad . \quad (29)$$

also that

$$P'_{m,n}Q_{m,n}-P_{m,n}Q'_{m,n}=\frac{(2n-1-2m)\ldots(1-2m)}{(2n+1+2m)\ldots(3+2m)}\,\frac{P_{m,1}Q_{m,0}-P_{m,0}Q_{m,1}}{2S} \quad . \quad . \quad (30)$$

13. In the same way as relations have been found between successive orders of toroidal functions, relations may be found between successive ranks of

Not putting the order n in evidence, write

$$\psi_{m+1}=f\psi_m+\phi\psi'_m$$

Proceeding as before it will be found that f, ϕ must satisfy the equations

$$\left.\begin{aligned} &f''-\frac{2m+1}{S^2}f-\frac{4m^2-1}{2}\frac{C}{S^3}\phi+2\left(n^2+\frac{4m^2-1}{4S^2}\right)\phi'=0\\ &\phi''-\frac{2m+1}{S^2}\phi+2f'=0 \end{aligned}\right\}$$

which are satisfied by

$$\phi=A, \quad f=-\frac{2m+1}{2}A\frac{C}{S}$$

leading to the relations

$$P_{m+1}=A_m\left(P'_m-m\frac{C}{S}P_m\right)$$

In precisely the same manner it may be shown that

$$P_{m-1}=B_m\left(P'_m+m\frac{C}{S}P_m\right)$$

and that

$$A_mB_{m+1}=\frac{4}{4n^2-(2m+1)^2}$$

If we put

$$A_m=\frac{2}{2n+2m+1}$$

then

$$B_m=\frac{2}{2n-2m+1}$$

and when $m=0$ the $P_{0,n}$ have the same values as for the toroidal functions already discussed.

Finally then,

$$\left.\begin{aligned} 2SP'_{m,n} &= \quad 2mCP_{m,n}+(2n+1+2m)SP_{m+1,n} \\ 2SP'_{m,n} &= -2mCP_{m,n}+(2n+1-2m)SP_{m-1,n} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (31)$$

from which the sequence equation follows at once

$$(2m+2n+1)SP_{m+1}+4mCP_m+(2m-2n-1)SP_{m-1}=0 \quad . \quad . \quad . \quad . \quad (32)$$

If we write in this

$$P_m=\frac{2m(2m-2)\ldots 4.2}{(2m+1+2n)(2m-1+2n)\ldots(1+2n)}\left(\frac{C}{S}\right)^m u_m$$

then

$$u_{m+1}+2u_m+\frac{(2m-1)^2-4n^2}{(2m-1)^2-1}\cdot\left(\frac{S}{C}\right)^2 u_{m-1}=0$$

By combining the formulæ (26) and (31) it is also possible to obtain relations between order and rank together. For instance, from the first of equation (26) and the second of equation (31), we get

$$\begin{aligned} (2m+2n+1)P_{m,n+1} &= (2n+1)CP_{m,n}+2SP'_{m,n} \\ &= (2n+1)CP_{m,n}-2mCP_{m,n}+(2n+1-2m)SP_{m-1,n} \\ &= (2n+1-2m)(CP_{m,n}+SP_{m-1,n}) \end{aligned}$$

with three other relations.

The four formulæ are

$$\left.\begin{aligned} &(26\alpha,\ 31\alpha). && P_{m,n+1}-CP_{m,n}-SP_{m+1,n}=0 \\ &(26\alpha,\ 31\beta). && (2n+1+2m)\ P_{m,n+1}-(2n+1-2m)(CP_{m,n}+SP_{m-1,n})=0 \\ &(26\beta,\ 31\alpha). && (2m+1-2n)(P_{m,n-1}-CP_{m,n})-(2m+1+2n)SP_{m+1,n}=0 \\ &(26\beta,\ 31\beta). && (2m+1-2n)\ P_{m,n-1}+(2m-1+2n)CP_{m,n}+(2m-1-2n)SP_{m-1,n}=0 \end{aligned}\right\} \quad (32a)$$

We are now in a position to reduce still further the relations (29), (30).

For putting $n=0$ in the second of (32a)

$$(2m+1)P_{m,1}=-(2m-1)(CP_{m,0}+SP_{m-1,0})$$

whence

$$(2m+1)\{P_{m,1}Q_{m,0}-P_{m,0}Q_{m,1}\}=(2m-1)S(P_{m,0}Q_{m-1,0}-P_{m-1,0}Q_{m,0})$$

But (32)

$$(2m-1)\mathrm{SP}_{m.0}=-(2m-3)\mathrm{SP}_{m-2.0}-4(m-1)\mathrm{CP}_{m-1.0}$$

Therefore the above

$$=\mathrm{S}\{\mathrm{P}_{1.0}\mathrm{Q}_{0.0}-\mathrm{P}_{0.0}\mathrm{Q}_{1.0}\}$$

But from the first of (32a)

$$\mathrm{S}(\mathrm{P}_{1.0}\mathrm{Q}_{0.0}-\mathrm{P}_{0.0}\mathrm{Q}_{1.0})=\mathrm{P}_{0.1}\mathrm{Q}_{0.0}-\mathrm{P}_{0.0}\mathrm{Q}_{0.1}=2\pi$$

Hence

$$\left.\begin{aligned}\mathrm{P}_{m.n+1}\mathrm{Q}_{m.n}-\mathrm{P}_{m.n}\mathrm{Q}_{m.n+1}&=2\frac{(2n-1-2m)(2n-3-2m)\ldots(1-2m)}{(2n+1+2m)(2n-1+2m)\ldots(1+2m)}\cdot\pi\\ \text{and}\qquad \mathrm{P}'_{m.n}\mathrm{Q}_{m.n}-\mathrm{P}_{m.n}\mathrm{Q}'_{m.n}&=\frac{(2n-1-2m)\ldots(1-2m)}{(2n+1+2m)\ldots(1+2m)}\cdot\frac{\pi}{\mathrm{S}}\end{aligned}\right\}\quad(33)$$

In the same way, or by substituting in $\mathrm{P}'_{m.n}\mathrm{Q}_{m.n}-\mathrm{P}_{m.n}\mathrm{Q}'_{m.n}$ the first of (26) or the first of (31), there follows

$$\mathrm{P}_{m.n+1}\mathrm{Q}_{m.n}-\mathrm{P}_{m.n}\mathrm{Q}_{m.n+1}=\mathrm{S}(\mathrm{P}_{m+1.n}\mathrm{Q}_{m.n}-\mathrm{P}_{m.n}\mathrm{Q}_{m+1.n})$$

14. From the formulæ now developed it is possible to find the complete integral of the general differential equation. But as in applications the co-efficients are determined in terms of definite integrals it will be well also to consider the solutions from a different point of view. If any potential function be expanded in a series of multiple sines and cosines of v, w, multiplied by $\sqrt{\mathrm{C}-c}$, we know that the co-efficients must be of the form $\mathrm{AP}+\mathrm{BQ}$. Now such a function is the inverse distance of any point from a fixed point. Let us choose as fixed point, to simplify the expression as much as possible, a point on the axis of x within the critical circle, say $(u'.\pi.0)$. Then the distance of $(u.v.w)$ from this is

$$\frac{a\sqrt{2}}{\sqrt{(\mathrm{C}-c)(\mathrm{C}'+1)}}\{\mathrm{CC}'+c-\mathrm{SS}'\cos w\}^{\frac{1}{2}}$$

Hence if $\dfrac{\sqrt{\mathrm{C}'+1}}{\sqrt{\{\mathrm{CC}'+c-\mathrm{SS}'\cos w\}}}$ be expanded in a series, the coefficient of $\cos mw\cos nv$ will be of the form $\mathrm{AP}_{m.n}+\mathrm{BQ}_{m.n}$. Further, for points within the tore u' (*i.e.*, $u>u'$) $\mathrm{A}=0$, whilst for points without, and therefore including the axis ($u<u'$), $\mathrm{B}=0$.

Hence

$$\sqrt{\mathrm{C}'+1}\int_0^{2\pi}\int_0^{2\pi}\frac{\cos mw\cos nvdwdv}{\sqrt{\{\mathrm{CC}'+\cos v-\mathrm{SS}'\cos w\}}}=\mathrm{AP}_{m.n}\text{ or }\mathrm{BQ}_{m.n}$$

according as $u\lesseqgtr u'$.

Now if the fixed point be on the critical circle B is always equal to zero and $(\mathrm{C}'=\mathrm{S}'=\infty)$

$$AP_{m,n}=\int_0^{2\pi}\int_0^{2\pi}\frac{\cos mv\cos nv}{\sqrt{\{C-S\cos w\}}}\,dwdv$$

Here $A=0$ unless $n=0$. Hence

$$AP_{m,0}=4\pi\int_0^{\pi}\frac{\cos m\theta}{\sqrt{C-S\cos\theta}}\,d\theta$$

We have already found that

$$P_{0,n}\propto\int_0^{\pi}\frac{d\theta}{\{C-S\cos\theta\}^{\frac{2n+1}{2}}}$$

we are therefore led to expect that in general

$$P_{m,n}\propto\int_0^{\pi}\frac{\cos m\theta d\theta}{\{C-S\cos\theta\}^{\frac{2n+1}{2}}}$$

which can easily be shown to be the case.

By taking the fixed point at the origin we have

$$BQ_{m,n}\propto\int_0^{\pi}\int_0^{\pi}\frac{\cos mv\cos nvdwdv}{\sqrt{C-\cos v}}$$

Here $B=0$ unless $m=0$, and then

$$Q_{0,n}\propto\int_0^{\pi}\frac{\cos n\theta}{\sqrt{C-\cos\theta}}\,d\theta$$

an expression which has been already found.

These expressions as *single* definite integrals are already known to be solutions of the differential equations, and are given by HEINE in his 'Kugelfunctionen.' They may easily be proved directly, and connected with the values found already by the sequence equations, and the values for P_{00}, P_{01}, &c. Thus writing

$$P_{m,n}=A\int_0^{\pi}\frac{\cos m\theta d\theta}{(C-S\cos\theta)^{\frac{2n+1}{2}}}$$

the integral is easily shown to satisfy equations (32), and the only further condition requisite is that A shall be chosen so as to make it agree with $P_{0,n}$, $P_{0,n-1}$.

Now

$$P_{0,n}=\int_0^{\pi}\frac{d\theta}{(C-S\cos\theta)^{\frac{2n+1}{2}}}$$

Hence

$$A=1 \text{ and } P_{m,n}=\int_0^\pi \frac{\cos m\theta d\theta}{(C-S\cos\theta)^{\frac{2n+1}{2}}}$$

Returning to the general integral, since u, u' enter symmetrically, and since if $u \gtrless u'$, $u' \lessgtr u$, it follows that

$$\int_0^\pi\int_0^\pi \frac{\cos mw\cos nvdwdv}{\sqrt{CC'-\cos v-SS'\cos w}}=LP_{m,n}.Q'_{m,n} \text{ or } LP'_{m,n}.Q_{m,n}$$

according as $u \lessgtr u'$, where L is independent of u or u'. Hence L may be determined by giving particular values to u or u'. Suppose u' at the origin then

$$LQ_{m,n}=\lim_{u'=0}\frac{\int_0^\pi\int_0^\pi \frac{\cos mw\cos nvdwdv}{\sqrt{CC'-\cos v-SS'\cos w}}}{\int_0^\pi \frac{\cos m\theta d\theta}{(C'-S'\cos\theta)^{\frac{2n+1}{2}}}}$$

To find the value of this expand the expressions under the integrals in ascending powers of S′, which is ultimately to vanish.

Then if

$$p<m \quad \int_0^\pi \cos mw\cos^p w dw=0$$

$$p=m \quad \int_0^\pi \cos mw\cos^m w dw=\frac{\pi}{2^m}$$

$$p>m \quad \text{the integral is finite} = I \text{ (say)}$$

Hence

$$LQ_{m,n}=\lim\frac{\int_0^\pi \cos nvdv\int_0^\pi \frac{\cos mw}{\sqrt{CC'-\cos v}}\left\{\alpha_m\left(\frac{SS'\cos w}{CC'-\cos v}\right)^m+\ldots\right\}}{\frac{1}{C'^{\frac{2n+1}{2}}}\int_0^\pi \cos m\theta\left(\beta_m\frac{S'^m\cos^m\theta}{C'^m}+\ldots\right)}$$

where

$$\alpha_m=\text{co. of } x^m \text{ in } (1-x)^{-\frac{1}{2}}=\frac{1.3.5\ldots(2m-1)}{2^p\underline{m}}$$

$$\beta_m= \quad ,, \quad ,, \quad (1-x)^{\frac{2n+1}{-2}}=\frac{(2n+1)(2n+3)\ldots(2n+2m-1)}{2^p\underline{m}}$$

Hence

$$LQ_{m,n}=\frac{\alpha_m}{\beta_m}S^m\int_0^\pi \frac{\cos nvdv}{(C-\cos v)^{\frac{2n+1}{2}}}$$

This at once gives us an expression for $Q_{m,n}$, viz.:

$$Q_{m,n}=MS^m\int_0^\pi \frac{\cos nv\,dv}{(C-\cos v)^{\frac{2m+1}{2}}}$$

where M is some constant depending on $m.n$. This has now to be found. If we write

$$U_{m,n}=S^m\int_0^\pi \frac{\cos nv\,dv}{(C-\cos v)^{\frac{2m+1}{2}}}$$

it is easily shown that

$$(2n+1-2m)U_{m,n+1}-4nCU_{m,n}+(2n-1+2m)U_{m,n-1}=0$$

This will agree with (27) if

$$NU_{m,n}=\frac{(2m+2n-1)\ldots(2m+1)}{(2n-1-2m)\ldots(1-2m)}Q_{m,n}$$

Hence

$$N=\frac{(2m-1+2n)\ldots(2m+1)}{(2n-1-2m)\ldots(1-2m)}M$$

and

$$Q_{m,0}=NS^m\int\frac{dv}{(C-\cos v)^{\frac{2m+1}{2}}}$$

where N is a function of m only.

Here again this is found to satisfy

$$(2m+1)SQ_{m+1,0}-4mCQ_{m,0}+(2m-1)SQ_{m-1,0}=0$$

which agrees with (32) if

$$N=(-)^mN$$

and

$$Q_{0,0}=N\int_0\frac{dv}{\sqrt{C-\cos v}}$$

But from the known value of $Q_{0,0}$ we see that $N=1/\sqrt{2}$. Hence

$$Q_{m,n}=\frac{(-)^m}{\sqrt{2}}\frac{(2n-1-2m)\ldots(1-2m)}{(2n-1+2m)\ldots(1+2m)}S^m\int_0^\pi\frac{\cos n\theta d\theta}{(C-\cos\theta)^{\frac{2m+1}{2}}}\quad\ldots\quad(34)$$

Also

$$L=\frac{\alpha_m}{M\beta_m}$$
$$=(-)^m\sqrt{2}\frac{1.3.5\ldots(2m-1)}{(2n-1+2m)(2n-3+2m)\ldots(2n+1)}\cdot\frac{(2n-1+2m)\ldots(2m+1)}{(2n-1-2m)\ldots(1-2m)}$$
$$=\sqrt{2}L_{m,n}\quad\text{(say)}$$

Since the distance between two points is

$$\frac{a\sqrt{2}}{\sqrt{(C-c)(C'-c')}}\{CC'-\cos(v-v')-SS'\cos(w-w')\}^{\frac{1}{2}}$$

it follows that the potential for a unit point at $(u'.v'.w')$ is

$$\left.\begin{aligned}\phi&=\frac{1}{a}\sqrt{(C-c)(C'-c')}\Sigma\Sigma L_{m,n}P_{m,n}Q'_{m,n}\cos n(v-v')\cos m(w-w')\\ \text{for points outside the tore } u';\ &\text{whilst for points inside, it is}\\ \phi&=\frac{1}{a}\sqrt{(C-c)(C'-c')}\Sigma\Sigma L_{m,n}P'_{m,n}Q_{m,n}\cos n(v-v')\cos m(w-w')\end{aligned}\right\}\quad\ldots\quad(35)$$

where, when $m=0$ or $n=0$, half the above value for $L_{m,n}$ must be taken.

When $u=0$ $P_{m,n}=0$ except when $m=0$ when $P_{0,n}=\pi$, which agrees with the value found in section II.

Also $P_{m,n}$ behaves in a similar way to $P_{0,n}$ for increasing n, whilst $P_{m+1,n}>P_{m,n}$ when m is large, as is clear at once from the integral expression for $P_{m,n}$.

Also since

$$Q_{m,n}\propto S^m\int_0^\infty\frac{\cos n\theta d\theta}{(C-\cos\theta)^{\frac{2m+1}{2}}}$$

it is clear that when $u=\infty$ $Q_{m,n}=0$ for all values of $m.n$. Also $Q_{m,n}$ behaves as $Q_{0,n}$ for increasing n.

IV.

TORES WITH NO CENTRAL OPENING.

15. In the case where the hole of a tore vanishes the functions hitherto considered become nugatory. In this case we must have recourse to the co-ordinates already referred to in (8). It is not here intended to develop the theory with the fulness of the general case. The functional differential equation has been shown to be

$$\frac{d^2\psi}{du^2}-n^2\psi-\frac{4m^2-1}{4u^2}\psi=0$$

In this write $\psi=\sqrt{u}G$ when

$$\frac{d^2G}{du^2}+\frac{1}{u}\frac{dG}{du}-n^2G-\frac{m^2}{u^2}G=0$$

the equation of cylindric harmonics (BESSEL's functions) with imaginary argument. Let G.H be the two particular integrals corresponding to the cylindric function J.Y of the first and second kind. Then

$$G_m(nu)=J_m(nui)$$
$$H_m(nu)=Y_m(nui)$$

And the potential function can be expressed in the form

$$\phi=\sqrt{u^3}+v^2\Sigma\Sigma(A_mG_m(nu)+B_mH_m(nu))\cos(nv+\alpha)\cos(mw+\beta)$$

Many of the properties of these functions can be at once written down from the analogous properties of J.Y. Thus

$$G_m(nu)=\frac{(nu)^m}{2.4\ldots2m}\left\{1+\frac{n^2u^2}{2.2m+2}+\frac{n^4u^4}{2.4.(2m+2)(2m+4)}+\ldots\right\}$$
$$=\frac{1}{\pi}\int_0^\pi\cos(nu\sin\theta-m\theta)d\theta$$
$$=\frac{(nu)^m}{1.3.5\ldots2m-1}\cdot\frac{1}{\pi}\int_0^\pi\cos(nz\cos\theta)\sin^{2m}\theta d\theta$$
&c.

So also

$$u\frac{dG_m}{du}=mG_m-uG_{m+1}$$
$$=uG_{m-1}-mG_m$$

and

$$G_{m+1}-\frac{2m}{u}G_m+G_{m-1}=0$$

which equations the H also satisfy.

The sequence equation has been solved by LOMMELL,* so as to fully express G_m and H_m in terms of G_0, G_1, H_0, H_1. But in any particular case where the values are required it is best to calculate successively by means of the sequence equation direct.

In the space within a tore u can become infinite, viz.: at the origin, and is never zero; this is evident from the equation

$$u=\frac{a\rho}{\rho^2+z^2}$$

* 'Studien über die Bessel'schen Functionen,' p. 4.

Without, it may become zero along the axis but infinite nowhere, for as it approaches the origin u must approach a finite limit which depends on the circle along which it moves. Now when u is infinite, G is infinite. Hence the G functions belong to space outside a tore. We are led to conclude that the H functions belong to space within. This may be proved as follows: Amongst many integral expressions known for Y_0 one is given by HEINE,* viz.:

$$\int_0^\infty e^{iz\cosh\theta}d\theta$$

This suggests

$$H_0(u)=A\int_0^\infty e^{-u\cosh\theta}d\theta$$

This is easily verified, for substituting in the differential equation it has to be shown that

$$\int_0^\infty (\sinh^2\theta-\frac{1}{u}\cosh\theta)e^{-u\cosh\theta}d\theta=0$$

which, on integrating the first term by parts follows at once. From this form we gather that

when $u=0$ $\quad H_0=\int_0^\infty d\theta=\infty$

„ $u=\infty$ $\quad H_0=0$

whence H_0 is the proper function for space within a tore. From the sequence equation this is seen to apply also to the H_n in general.

V.

EXAMPLES AND APPLICATIONS.

In this section I propose to give a few examples of the application of the foregoing theory, to the solution of physical problems.

16. *Potential of a ring whose axis is the same as the critical circle.*

Let z' be its distance from the plane of the critical circle, b its radius, u', v' its dipolar co-ordinates.

Then the potential is

$$2\mu b\int_0^\pi \frac{d\theta}{\sqrt{(z-z')^2+\rho^2+b^2-2b\rho\cos\theta}}$$

This expanded takes the form

$$\sqrt{C-c}\Sigma A_n P_n\cos(nv+\alpha_n)$$

for points outside the tore u'.

* 'Kugelfunctionen,' p. 191.

For a point on the axis, $u=0$ $P_n=\pi$ and the above become respectively

$$\frac{2\mu\pi\sqrt{1-c}S'}{\sqrt{2(C'-\cos\overline{v-v'})}} \text{ and } \pi\sqrt{1-c}\Sigma A_n\cos(nv+\alpha_n).$$

It will therefore be more convenient to determine the A_n, α_n from this simplified case.

It is clear that $1/\{C'-\cos\overline{v-v'}\}^{\frac{1}{2}}$ can be expanded in a series of powers of cosines of $(v-v')$ and therefore of multiples of the same.

Hence

$$\alpha_n=-nv'$$

and

$$\frac{2\mu S'}{\sqrt{2(C'-\cos\theta)}}=\Sigma A_n\cos n\theta.$$

Therefore

$$\pi A_n=\frac{2\mu S'}{\sqrt{2}}\int_0^{2\pi}\frac{\cos n\theta d\theta}{\sqrt{C'-\cos\theta}}$$
$$=4\mu S'Q'_n$$

But

$$\pi A_0=2\mu S'Q'_0$$

Hence in general the potential for points outside the tore u' is

$$\phi=\frac{4\mu S'}{\pi}\left(\frac{C-c}{C'-c'}\right)^{\frac{1}{2}}\{\Sigma P_nQ'_n\cos n(v-v')-\tfrac{1}{2}P_0Q'_0\} \quad (36a)$$

Consequently the potential for points within the tore u' is

$$\phi=\frac{4\mu S'}{\pi}\left(\frac{C-c}{C'-c'}\right)^{\frac{1}{2}}\{\Sigma P'_nQ_n\cos n(v-v')-\tfrac{1}{2}P'_0Q_0\} \quad (36b)$$

Both these series have been shown to be convergent.

If M be the whole mass of the ring

$$M=2\pi b\mu=2\pi\mu\frac{aS'}{C'-c'}$$

It follows as a corollary that the potential for a mass M on the axis is, for all points not on the axis,

$$\frac{2M\sqrt{2}\sin\frac{v'}{2}}{a\pi}\sqrt{C-c}\{\Sigma_0^\infty Q_n\cos n(v-v')-\tfrac{1}{2}Q_0\} \quad (37)$$

Also, putting M at $0.v'$ and $-M$ at $0.-v'$, and making v' zero and M infinite, the potential for a uniform field of force parallel to axis is

$$\mu\sqrt{C-c}\Sigma_0^\infty nQ_n\sin nv \quad (38)$$

17. *Electric potential of a tore and its capacity.*

Let V be the constant potential of the tore (u'). Then (A_n, α_n) must be determined, so that

$$\phi=\sqrt{C-c}\Sigma_0^{\infty}A_nP_n\cos n(v+\alpha_n)$$

may=V for all values of v when $u=u'$.

Hence $\alpha_n=0$ and

$$\pi A_nP'_n=2V\int_0^{\pi}\frac{\cos n\theta d\theta}{\sqrt{C'-\cos\theta}}$$

$$=2\sqrt{2}VQ'_n$$

and

$$\pi A_0P'_0=\sqrt{2}VQ'_0$$

$$\therefore\ \phi=\frac{\sqrt{2}V}{\pi}\sqrt{C-c}\left\{2\Sigma_1\frac{Q'_n}{P'_n}P_n\cos nv+\frac{Q'_0}{P'_0}P_0\right\}\quad\ldots\ldots\quad(39)$$

This series is easily seen to be convergent, since (§ 10) it is less than $\Sigma\frac{(C+S)^n}{(C'+S')^{2n}}$, where $C+S<C'+S'$.

To find the capacity of the ring we must take the surface integral of $\frac{1}{4\pi V}\frac{\delta\phi}{\delta n}$ over it. So, q denoting the capacity,

$$q=\frac{1}{4\pi V}\int_0^{2\pi}2\pi\rho\frac{dn'}{dv}dv\frac{\delta\phi}{\delta u}\frac{du}{dn}$$

$$=\frac{\sqrt{2}aS}{\pi}\Sigma\int_0^{2\pi}\frac{1}{C-c}\left\{\frac{S}{2\sqrt{C-c}}P_n+\sqrt{C-c}\frac{dP_n}{du}\right\}\frac{Q'_n}{P'_n}\cos nvdv$$

or dropping the dashes, and writing

$$\frac{dP_n}{du}=\frac{2n+1}{2S}(P_{n+1}-CP_n)$$

$$q=\frac{2\sqrt{2}aSQ_n}{\pi}\int_0^{\pi}\left\{-\frac{d}{du}+\frac{2n+1}{2S}\left(\frac{P_{n+1}}{P_n}-C\right)\right\}\frac{\cos nv}{\sqrt{C-c}}dv$$

Now

$$\int_0^{\pi}\frac{\cos nvdv}{\sqrt{C-c}}=\sqrt{2}Q_n$$

and

$$\frac{dQ_n}{du}=\frac{2n+1}{2S}(Q_{n+1}-CQ_n)$$

whence

$$q=\frac{2a}{\pi}\Sigma(2n+1)(P_{n+1}Q_n-P_nQ_{n+1})\frac{Q_n}{P_n}$$

$$=4a\Sigma_1\frac{Q_n}{P_n}+2a\frac{Q_0}{P_0}\text{ by (24)}\quad\ldots\ldots\quad(40)$$

This expression for the capacity in an infinite series is more convergent than Σe^{-2nu}.

When the section of the ring is not very large compared with the radius of its circular axis,

$$\left.\begin{aligned} q &= 2a\left(\frac{Q_0}{P_0}+2\frac{Q_1}{P_1}\right) \text{ very nearly}^* \\ &= 2a\left(\frac{F'}{F}+2\frac{F'-E'}{E}\right) \\ \text{or} \quad &= 2\sqrt{R^2-r^2}\left\{3\frac{F'}{F}-\frac{\pi}{EF}\right\} \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (40a)$$

where

$$k^2=2\frac{\sqrt{R^2-r^2}}{R+\sqrt{R^2-r^2}}$$

Measured in terms of the capacity of a sphere whose radius is equal to a tangent from the centre to the tore, the capacity is

$$2\frac{3EF'-\pi}{EF}$$

When $R=3r$ the omission of the term depending on $\frac{Q_2}{P_2}$ introduces an error of about ·27 per cent.

k^2 may be expressed in terms of the angle subtended at the centre by the tore, viz.: if this angle be 2α,

$$k^2=\frac{2\cos\alpha}{1+\cos\alpha}=\cos\alpha\sec^2\frac{\alpha}{2}$$

$$k'=\tan\frac{\alpha}{2}$$

When

$k'=\sin 3^0$ (about $r=\frac{1}{10}R$) $\quad q=$ ·733 × capacity of above sphere

$k'=\sin 6^0$ (about $r=\frac{1}{5}R$) $\quad q=$ ·898 × . . .

18. We may find the potential also for the electricity induced on a tore, put to earth, by a charged circular wire with the same axis as the tore. For the potential of the wire (u', v') for points within (u') is (36b)

$$\phi_1=\frac{4\mu S'}{\pi}\sqrt{C-c}\{\Sigma P'_n Q_n \cos n(v-v')-\tfrac{1}{2}P'_0Q_0\}$$

whilst that for points outside the tore (u_0) due to the charge induced on it is

$$\phi_2=\sqrt{C-c}\Sigma A_n P'_n \cos n(v-\alpha_n)$$

* The expression given in the 'Proceedings' is incorrect.

and the condition is that when $u=u_0$ $\phi_1+\phi_2=0$

$$\therefore \alpha_n=v',\ A_nP^0{}_n=-\frac{4\mu S'}{\pi}P'_nQ^0{}_n,\text{ and } A_0P^0{}_0=-\frac{2\mu S'}{\pi}P'_0Q^0{}_0$$

Whence

$$\phi=\frac{2\mu S'}{\pi}\sqrt{C-c}\Sigma\left\{2\frac{P'_n}{P^0_n}(P^0{}_nQ_n-Q^0{}_nP_n)\cos n(v-v')+\frac{P'_0}{P^0_0}(P^0{}_0Q_0-Q^0{}_0P_0)\right\}\ .\quad (41)$$

and the general solution when the tore is insulated and has a charge of its own is found by adding the potential found in the last article.

Also if the section of the wire be very small we can find the capacity of the system approximately, by supposing the wire to coincide with one of the equipotential surfaces near it.

19. As an example of the use of tesseral functions with constant surface conditions, we will take the problem of the electrical induction on a tore under the influence of a point arbitrarily placed. We lose no generality by supposing it in the plane of (xz); let then its co-ordinates be $(u'.v'.0)$. The potential due to this for points within u' has been found at the end of Section III., viz.,

$$\phi=\frac{\mu}{a}\sqrt{(C-c)(C'-c')}\,\Sigma L_{m,n}P'_{m,n}Q_{m,n}\cos mw\cos n(v-v')$$

As before, the potential of the induced charge will be of the form

$$\phi=\sqrt{C-c}\,\Sigma A_{m,n}P_{m,n}\cos mw\cos n(v-v')$$

and (the tore being u_0)

$$A_{m,n}P^0{}_{m,n}=-\mu\frac{\sqrt{C'-c'}}{a}L_{m,n}P'_{m,n}Q^0{}_{m,n}$$

$$\phi=\mu\frac{\sqrt{(C-c)(C'-c')}}{a}\Sigma L_{m,n}\frac{P'_{m,n}}{P^0_{m,n}}(P^0{}_{m,n}Q_{m,n}-Q^0{}_{m,n}P_{m,n})\cos mw\cos n(v-v')\ .\quad (42)$$

When the point is on the axis, all these terms vanish (§ 14) except for $m=0$.

If necessary, also, the capacity of a tore and a very small sphere can be found approximately from these formulæ.

20. One more example illustrating the application to cases of differential surface conditions may be given. Take the case of a tore moving parallel to its axis through an infinite fluid with velocity V. Here the conditions are that if ϕ be the velocity potential for fluid moving past it,

$$\phi=-Vz+\phi_1$$

and

$$\frac{\delta\phi}{\delta u}=0 \text{ when } u=u_0.$$

The expansion for z has already been given, viz.: (for points not on the axis)

$$\mu\sqrt{C-c}\,\Sigma_0^\infty nQ_n \sin nv$$

To determine μ we notice that at the critical circle (as everywhere on the plane of xy)

$$\frac{d\phi}{dv}\frac{dv}{dn}=-V$$

Taking a point outside the critical circle

$$-V=\mu\frac{(C-1)^{\frac{1}{2}}}{a}\Sigma n^2Q_n$$

The easiest way to calculate this is to make the point approach the critical circle, *i.e.*, $u=\infty$, when

$$-V=\frac{\mu}{a}\lim(C-1)^{\frac{1}{2}}Q_1$$

$$=\frac{\mu}{a}\lim(C-1)^{\frac{1}{2}}\int_0^\infty\frac{d\theta}{(C+S\cosh\theta)^{\frac{1}{2}}}$$

$$=\frac{\mu}{a}\int_0^\infty\frac{d\theta}{2^{\frac{1}{2}}\cosh^3\frac{\theta}{2}}=\frac{\mu}{a}\cdot\frac{\pi}{4\sqrt{2}}$$

which gives the theorem

$$\Sigma n^2Q_n=\frac{\pi}{4\sqrt{2}}(C-1)^{-\frac{1}{2}}=\frac{\pi}{16}\operatorname{cosech}^3\frac{u}{2}$$

Hence

$$\phi=\frac{4a\sqrt{2}}{\pi}V\sqrt{C-c}\Sigma(A_nP_n\sin n(v-\alpha_n)-nQ_n\sin nv)$$

where $\frac{\delta\phi}{\delta u}=0$ when $u=u_0$ for all values of v. The terms in cos nv would merely increase ϕ by the series for a constant, we may therefore without loss of generality put $\alpha_n=0$, and then, using dashed letters to denote differential coefficients,

$$\frac{1}{4\sqrt{2}}\frac{\pi}{a}\frac{\delta\phi}{\delta u}=\Sigma\left\{\frac{S}{2\sqrt{C-c}}(A_nP_n-nQ_n)+\sqrt{C-c}(A_nP'_n-nQ'_n)\right\}\sin nv$$
$$=0 \text{ when } u=u_0$$

$$\therefore\ \Sigma\{S(A_nP_n-nQ_n)+2(C-c)(A_nP'_n-nQ'_n)\}\sin nv=0$$

$$A_{n+1}P'_{n+1}+A'_{n-1}P'_{n-1}-A_n(SP_n+2CP'_n)$$
$$=(n+1)Q'_{n+1}+(n-1)Q'_{n-1}-n(SQ_n+2CQ'_n$$

Now it is easily shown that

$$P'_{n+1}+P'_{n-1}-(SP_n+2CP'_n)=0$$

with a similar formula for Q_n, we may hence write the above equation

$$(A_{n+1}-A_n)P'_{n+1}-(A_n-A_{n-1})P'_{n-1}=Q'_{n+1}-Q'_{n-1}$$

with initial equation

$$(A_2-A_1)P'_2-A_1P'_0=Q'_2-Q'_0$$

To get a first integral of this write the successive equations in order, multiply those containing P'_{r+1}, P'_{r-1} by P'_r and add, we get

$$(A_{n+1}-A_n)P'_{n+1}P'_n-A_1P'_1P'_0=P'_nQ'_{n+1}-P'_1Q'_0+\Sigma_1^{n-1}(P'_rQ'_{r+1}-P'_{r+1}Q'_r)$$

$$=P'_nQ'_{n+1}-P'_0Q'_1+\frac{\pi}{2}\Sigma_0^{n-1}(2r+1)$$

$$\therefore\ A_{n+1}-A_n=\frac{Q'_{n+1}}{P'_{n+1}}+\frac{(A_1P'_1-Q'_1)P'_0}{P'_nP'_{n+1}}+\frac{\pi}{2}\cdot\frac{n^2}{P'_nP'_{n+1}}$$

Put

$$(A_1P'_1-Q'_1)P'_0=\frac{\pi}{2}\alpha,\quad \frac{Q'_n}{P'_n}=x_n,$$

then since

$$\frac{\pi}{2}=\frac{1}{2n+1}(P'_nQ'_{n+1}-P'_{n+1}Q'_n)$$

$$A_{n+1}-A_n=x_{n+1}+\frac{n^2+\alpha}{2n+1}(x_{n+1}-x_n)$$

$$=\frac{1}{2n+1}\{(n+1)^2+\alpha)x_{n+1}-(n^2+\alpha)x_n\}$$

Hence

$$A_{n+1}-A_1=\frac{(n+1)^2+\alpha}{2n+1}x_{n+1}+2\Sigma_2^n\frac{r^2+\alpha}{4r^2-1}x_r-\frac{1+\alpha}{3}x_1$$

and

$$A_{n+1}=\frac{(n+1)^2+\alpha}{2n+1}x_{n+1}+2\Sigma_1^n\frac{r^2+\alpha}{4r^2-1}x_r-\alpha x_0$$

A_n is undetermined to the extent of α; but since the velocity potential must be finite everywhere, α must be chosen so that the series $\Sigma A_nP_n(u)$ shall be convergent. It will first be necessary to prove that A_n is finite when n is large; α must then be chosen so that A_n vanishes for n infinite, and lastly, it will remain to show that with

this value of α the series $\Sigma A_n P_n(u)$ is convergent, from which the convergency of ϕ will flow at once. Now

$$-x_n = -\frac{Q'_n}{P'_n} = \frac{Q_{n-1} - CQ_n}{CP_n - P_{n-1}}$$

$$< \frac{Q_{n-1} - CQ_n}{S^2 P_{n-1}}$$

$$\therefore -\Sigma_1^\infty x_n < \frac{1}{S^2}\Sigma_1^\infty \frac{Q_{n-1}}{P_{n-1}} - \frac{C}{S^2}\Sigma_1^\infty \frac{Q_n}{P_{n-1}}$$

Both the series on the right are finite, hence so also are $-\Sigma_1^\infty x_r$ and $\Sigma_1^\infty \frac{r^2+\alpha}{4r^2-1}x_r$, and A_n tends to a finite limit with increasing n. It is therefore possible to give α a value which shall make this limit zero. It is given by

$$2\Sigma_1^\infty \frac{r^2+\alpha}{4r^2-1}x_r - \alpha x_0 = 0$$

whence

$$A_n = \frac{n^2+\alpha}{2n-1}x_n - 2\Sigma_n^\infty \frac{r^2+\alpha}{4r^2-1}x_r \quad \ldots\ldots\ldots \quad (43)$$

Lastly it remains to consider the convergency of the series $\Sigma A_n P_n(u)$. When n is very large $-A_{n+1}$ tends to the limit $-\frac{(n+1)^2+\alpha}{2n+1}x_{n+1}$ which is

$$< \frac{(n+1)^2+\alpha}{(2n+1)S^2}\left(\frac{Q_{n-1}}{P_{n-1}} - C\frac{Q_n}{P_{n-1}}\right)$$

Also since $u < u_0$ $P_n(u) < P_n$. Hence the series under consideration is

$$< \frac{1}{S^2}\Sigma \frac{n^2+\alpha}{(2n-1)}\left\{\frac{Q_{n-2}P_n}{P_{n-2}} - C\frac{Q_{n-1}P_n}{P_{n-2}}\right\}$$

The sum of the first set of terms is $< \frac{(C+S)^2}{S^2}\Sigma \frac{n^2+\alpha}{2n-1}Q_{n-2}$, and of the second set is $< \frac{C(C+S)^2}{S^2}\Sigma \frac{n^2+\alpha}{2n-1}Q_{n-1}$; both of these are finite. Hence the sum $\Sigma A_n P_n$ is finite and *à fortiori* the sum $\Sigma A_n P_n(u) \sin nv$.

Finally then the velocity potential for fluid motion due to a tore moving parallel to itself through a fluid at rest at infinity is

$$\phi = \frac{4a\sqrt{2}}{\pi} V \sqrt{C-c}\,\Sigma_1^\infty A_n P_n \sin nv$$

where A_n is given by (43) and

$$\alpha=\frac{-2\Sigma_1^{\infty}\frac{\tau^2}{4\tau^2-1}\frac{Q'_\tau}{P'_\tau}}{-x_0+2\Sigma_1^{\infty}\frac{1}{4\tau^2-1}\cdot\frac{Q'_\tau}{P'_\tau}}$$

and where $P'_\tau . Q'_\tau$ stand for $\frac{dP_\tau(u_0)}{du_0}, \frac{dQ_\tau(u_0)}{du_0}$

We may now find the energy of the fluid motion. This is, the density of the fluid being unity,

$$T=-\int_0^{2\pi}2\pi\rho\frac{dn'}{dv}dv\,.\,\phi\frac{d\phi}{dn}$$

and

$$\frac{dn'}{d\rho}=\frac{a}{C-c}$$

$$\frac{d\phi}{dn}=V\frac{z}{r}=V\frac{Ss}{C-c}$$

$$\rho=\frac{aS}{C-c}$$

$$\therefore T=-2\pi a^2VS^2\int_0^{2\pi}\frac{s}{(C-c)^3}\phi dv$$

$$=-8\sqrt{2}a^3V^2S^2\Sigma A_nP_n\int_0^{2\pi}\frac{\sin v\sin nv}{(C-c)^{\frac{1}{2}}}dv$$

$$=-\frac{16\sqrt{2}}{3}a^3V^2S\Sigma A_nP_n\frac{d}{du}\left(\frac{1}{S}\frac{d}{du}\right)\int_0^{2\pi}\frac{\cos(n-1)v-\cos(n+1)v}{\sqrt{C-c}}dv$$

But

$$\int_0^{2\pi}\frac{\cos nv}{\sqrt{C-c}}dv=2\sqrt{2}Q_n$$

$$\therefore T=-\frac{64}{3}a^3V^2S\Sigma A_nP_n\frac{d}{du}\frac{1}{S}(Q'_{n-1}-Q'_{n+1})$$

But

$$Q'_{n-1}-Q'_{n+1}=-2nSQ_n$$

$$\therefore T=\frac{128}{3}a^3V^2S\Sigma_1^{\infty}nA_nP_nQ'_n \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (44)$$

which is more convergent than the series for ϕ.

In a similar manner may be found the velocity potential for any motion of translation, or the magnetism induced in a uniform field of force.

[*September*, 1881.—At the suggestion of one of the Referees I give a few additional numerical illustrations. The first is the ratio of the density of electricity at a point on a tore furthest from the axis to that at a point nearest the axis. The potential due to the distribution of electricity on the tore is given by (39). The normal force at any point of the tore is

$$\frac{\delta\phi}{\delta u}\frac{\delta u}{\delta n}=-\frac{C-c}{a}\frac{\delta\phi}{\delta u}$$

whilst for points furthest from the axis $u=u'$, $v=0$, and for points nearest $u=v'$, $v=\pi$. Putting these in, remembering that $2SdP_n/du=(2n+1)(P_{n+1}-CP_n)$ and $(2n+1)(P_{n+1}Q_n-P_nQ_{n+1})=2\pi$, it is easily shown that the above ratio is

$$=\left(\frac{C+1}{C-1}\right)^{\frac{1}{2}}\frac{2\pi\left(\frac{1}{2P_0}-\frac{1}{P_1}+\frac{1}{P_2}-\cdots\right)-\frac{1}{2}(Q_0+Q_1)-2(C+1)\Sigma(-)^nnQ_n}{2\pi\left(\frac{1}{2P_0}+\frac{1}{P_1}+\frac{1}{P_2}+\cdots\right)+\frac{1}{2}(Q_0-Q_1)-2(C-1)\Sigma nQ_n}$$

If the first n sequence equations in Q be added together there results

$$4(C-1)\Sigma_0^n nQ_n=(2n+1)(Q_{n+1}-Q_n)+Q_0-Q_1$$

whence $4(C-1)\Sigma nQ_n=Q_0-Q_1$

Further, putting $(-)^nQ_n=q_n$, the sequence equation for q is

$$(2n+1)q_{n+1}+4nCq_n+(2n-1)q_{n-1}=0$$

whence as before

$$4(C+1)\Sigma(-)^nnQ_n=4(C+1)\Sigma nq_n=q_1-q_0=-(Q_0+Q_1)$$

Finally then the ratio of the densities is

$$=\left(\frac{1+k'}{1-k'}\right)^{\frac{1}{2}}\frac{\frac{1}{2P_0}+\Sigma_1\frac{(-)^n}{P_n}}{\frac{1}{2P_0}+\Sigma_1\frac{1}{P_n}}$$

If terms higher than P_1 be neglected this is

$$=\left(\frac{1+k'}{1-k'}\right)^{\frac{1}{2}}\frac{E-2k'F}{E+2k'F}$$

I have not been able to find a finite expression for $\Sigma 1/P_n$ and $\Sigma(-)^n/P_n$, but when the ratio of r to R is very small, the first two terms are sufficient. In any other case

we can easily find the limits of error produced by neglecting terms after a given one. Thus suppose all after P_r be neglected, then § 10

$$\frac{1}{P_n} > \frac{1}{C+S}\frac{1}{P_{n-1}} < \frac{1}{CP_{n-1}}$$

whence it follows that

$$\Sigma_{r+1}^{n} 1/P_n > \frac{k'}{1-k'}\frac{1}{P_r} < \frac{2k'}{(1-k')^2}\frac{1}{P_r}$$

Similarly it may be shown that, r being odd

$$\Sigma_{r+1}^{n}(-)^{n}/P_n > \frac{k'}{1+k'} - \frac{2k'^2}{(1-k')^2}\frac{1}{P_r}$$

$$< \frac{2k'}{1+k'^2}\left\{\left(\frac{1+k'^2}{1-k'^2}\right)^2 - \frac{k'}{1-k'^2}\right\}\frac{1}{P_r}$$

For the two cases of $k' = \sin 3°$ and $k' = \sin 6°$ (corresponding very nearly to $R = 10r$ and $5r$ respectively) the ratios are ·5171 and ·2656.

The ratio of the velocity of the fluid at the centre of a tore to that of the tore itself when it moves without cyclic motion, parallel to its axis, is easily found. The point is given by $u=0$ $v=\pi$ which makes $P_n=\pi$. The velocity of the fluid $=\frac{\delta\phi}{\delta v}\frac{dv}{dn}$

$$= \frac{8V\sqrt{2}}{\pi}\sqrt{2}\Sigma_1^n n A_n \pi(-)^n$$

therefore ratio $= -16\Sigma_1^n(-)^n n A_n$

In the table below are given the values in two cases of α, A_1, A_2, T′ (the effective mass of the fluid measured in terms of the fluid displaced), and V′, the ratio of the velocity at the centre, to that at an infinite distance when the tore is held at rest in the stream.

k'	α	A_1	A_2	T′	V′
sin 3°	−·00645	−·00216	−·00000	·99995	1·03456
sin 6°	−·01868	−·00871	−·00007	1·09449	1·13712

Suppose the tore held in a uniform field of electric force parallel to its axis. The potential of the field is

$$\phi = \mu z$$

$$= \frac{4a\mu\sqrt{2}}{\pi}\sqrt{(C-c)}\Sigma n Q_n \sin nv$$

Hence, supposing the tore to be at zero potential and to have no charge, the potential of the disturbed field is, dashed letters denoting functions of u',

$$=-\frac{4a\mu\sqrt{2}}{\pi}\sqrt{(C-c)}\Sigma n\left(Q_n-\frac{Q'_n}{P'_n}P_n\right)\sin nv$$

The density at any point of the tore is

$$=-\frac{\mu\sqrt{2}}{\pi^2}(C'-c)^{\frac{1}{2}}\Sigma\frac{n}{P'_n}\left\{P'_n\frac{dQ'_n}{du}-Q'_n\frac{dP'_n}{du}\right\}\sin nv$$

$$=-\frac{\mu\sqrt{2}}{\pi S'}(C-c)^{\frac{1}{2}}\Sigma\frac{n^2}{P'_n}\sin nv$$

Now at the points where the osculating plane touches the tore $z=r$ and $\rho=R$, whence

$$C-c=Ss \text{ or } c=1/C \qquad s=S/C$$

The greatest density on a sphere similarly influenced is $\frac{3\mu}{4\pi}$. The ratio is then

$$=\frac{4\sqrt{2}}{3}\frac{S^2}{C^3}\left\{\frac{S}{CP'_1}+\frac{4S}{C^2P'_2}+\right\}$$

The value of this ratio for the cases already considered are

for $k'=\sin 3°$, ·675

„ $k'=\sin 6°$, ·698

When the direction of the electric field is perpendicular to the axis, its potential is

$$\phi_1=\mu\rho\cos w=\mu a\frac{S\cos w}{C-c}$$

Hence clearly the functions for the expansion of this are the tesseral functions, P_{1n}, Q_{1n}, and the conditions, since the potential holds for space outside the tore, are that

$$\mu a\frac{S\cos w}{C-c}+\sqrt{C-c}\cos w\Sigma A_nP_{1n}\cos nv=0$$

when $u=u'$ for all values of v.

Hence

$$\pi A_n P'_{1.n} + 2\mu a S \int_0^{\pi} \frac{\cos nv}{(C-c)^{\frac{3}{2}}}\, dv = 0$$

or

$$\pi A_n P'_{1.n} - 4\mu a \frac{d}{du}(Q'_{0.n}\sqrt{2}) = 0$$

and

$$A_n = \frac{2\mu a\sqrt{2}}{\pi S P'_{1.n}}(2n+1)(Q'_{0.n+1} - CQ'_{0.n})$$

But

$$A_0 = \frac{\mu a\sqrt{2}}{\mu S P_{1.0}}(Q'_{0.1} - CQ'_{0.0})$$

From the first of (32*a*)

$$SP_{1.n} = P_{0.n+1} - CP_{0.n} = \frac{2S}{2n+1}\frac{d}{du}(P_{0.n})$$

which enables us to write the above in several ways. As before, the densities of electricity induced at points $(u'.0.0)$ and $(u'.\pi.0)$ are easily found.]

XV. Polacanthus Foxii, *a large undescribed Dinosaur from the Wealden Formation in the Isle of Wight.*

By J. W. Hulke, *F.R.S.*

Received January 3,—Read January 27, 1881.

[Plates 70–76.]

For the opportunity of studying the remains described in this note I am indebted to the courtesy of the Rev. W. Fox, of Brixton, Isle of Wight, who last autumn gave me free access to his rich collection of fossils obtained in that locality.

Much shattered by being very hastily dug out, and since much damaged by the accidental breakages and the dissociations scarcely avoidable in the absence of a suitable place for their safe-keeping, there is risk of these remains becoming before long lost to the palæontologist. In view of this not improbable eventuality I venture to offer to the Royal Society these notes, in writing which I have been reminded that it was to this Society the late Dr. G. A. Mantell, now more than fifty years since, communicated his first discoveries of Iguanodont and Hylæosaurian remains.

The remains of *Polacanthus* were found by Mr. Fox in 1865 in a bed of blue shaly clay, which occurs near the middle of the cliff, a short distance east of Barne's Chine. The bed is easily recognisable by the large quantities of lignite which it contains.

Professor R. Owen, to whom Mr. Fox showed some of these fossils soon after their discovery, suggested for the animal indicated by them the name *Polacanthus*—many-spined—*P. Foxii*, and this name Mr. Fox adopted in an account of his discovery read by him at the next meeting of the British Association. A brief notice of the discovery with a rude woodcut also appeared about the same time in the "Illustrated London News." Both these communications have only the value of preliminary notices by persons without anatomical training, and no description of the fossils sufficient for the use of palæontologists has yet appeared.

Mr. Fox's MS., read at the meeting of the British Association, cannot now be found, and his paper does not appear in the "Reports." An abstract which I made of it in 1869 gives the following list of the parts he believed he had secured.

"Sacrum and pelvis; 7 lumbar, 7 anterior dorsal vertebræ with their ribs; 20 caudal vertebræ; 2 femora; 1 tibia with fibula; 3 metatarsals, phalanges, and 3 unguals; 20 to 30 large dermal spines, and as many scutes."

The scattered remains which last autumn I succeeded in bringing together again do

not quite agree with these numbers; the vertebræ, foot-bones, and dermal spines are fewer, and I was not able to identify any portions of Ilium, Ischium, or Pubis.

Vertebral column.—This is now represented by 11 præsacral, five sacral, and 15 postsacral or caudal vertebræ.

The 11 præsacral vertebræ comprise six disconnected and five anchylosed in a continuous series.

Of the six disconnected vertebræ three are fairly complete (Plate 70, figs. 1, 2). The form of the centrum is cylindroid; it is long relatively to its breadth, slightly constricted at its middle and expanded at its articular ends, which are plane or very slightly concave, the concavity of the posterior surface being most evident. The antero-posterior extent of the neurapophyses at their attachment to the centrum nearly equals the length of this latter. Their anterior margin rises nearly vertically from the centrum, whilst their posterior margin has a strong forward slant. The spinous processes of all the præsacral vertebræ are broken off and missing. The transverse processes in this series show a double costal articulation. In the level of the crown of the arch is a large, conspicuous, capitular, costal facet borne jointly by the arch and root of transverse process. It is directed outwards, and against it in two instances the rib-head, of an expanded discoid form, still abuts. Above this, the process, slender and trihedral in cross-section, is prolonged outwards and upwards above the rib-neck. It bore at its free end, as is shown by detached pieces, an articular surface for the tubercle of the rib. The length of the vertebral centrum (represented in Plate 70, figs. 1, 2) is 75 millims.; the horizontal diameter at the articular ends is 52 millims., and the vertical diameter here 51 millims.; and the horizontal diameter at the middle of the centrum is 35 millims. The double costal articulation places these six disconnected vertebræ in the front of the trunk. Between them and the five anchylosed vertebræ several are doubtless missing, since these last are demonstrated to belong to the loins by the anchylosis of the hindmost of the series to the first sacral centrum.

The lumbar centra (Plate 71, fig. 1) have a more attenuate form than that of those referred to the front of the chest. Their lateral surfaces slope inwards and meet somewhat angularly below. This is very apparent in the second in the chain. The spinous and transverse processes of all are broken off and missing.

The average length of the centrum in this series is 75 millims., the horizontal diameter at the middle is 33 millims., 30 millims., 30 millims., 34 millims.; and that of the articular ends is 40 millims., 37 millims., 36 millims.. 50 millims.

Sacrum (Plate 71, fig. 1).—This has the usual dinosaurian structure; it consists of a chain of five anchylosed centra much larger than the slender lumbar centra. Accidental cross-sections made by fractures show the form of the centrum to be remarkably depressed (fig. 2); it is, however, not improbable that this great excess of the horizontal over the vertical diameter may have been increased by pressure. These diameters are in the second centrum 70 millims. and 25 millims. respectively. The

under surface of the centrum is cylindroid; in the three foremost centra a shallow median groove indents it longitudinally. The piers of the neural arches rest each on two centra, and the sacral nerves escaped from the vertebral canal across the middle of a centrum except the last nerve, which passed out intervertebrally between the last sacral and the first caudal vertebra. A stout lower transverse process stands out from the side of the sacrum along the line of junction of each two centra, and doubtless the outer ends of these processes coalesced in the usual looplike manner: all have been broken off, and are now missing. The upper transverse processes which project from the crown of the arch are small and inconspicuous; they form, with the expanded summit of a dwarfed spinous process, the support of a stout dermal armour to be presently described. Comparison of the following measurements of the sacral centra with those of the præsacral centra already given will make the greater bulk of the former very apparent.

Sacral centra.	No. 1.	No. 2.	No. 3.	No. 4.	No. 5.
Horizontal diameter at the articular ends . . .	*	87	87	87	87
" " middle of centrum . . .	70	70	70	70†	82
Length of centrum	67	67	67	62	58

Post-sacral vertebræ (Plate 72, figs. 1, 2; Plate 73, figs. 1, 2; Plate 75, figs. 3, 4).—The 13 now remaining are certainly not a continuous series—many intermediate ones are missing; most are from the root of the tail.

The first caudal, known to be such by its exactly fitting the last sacral, differs from this most obviously by the centrum bearing entirely its own arch. It and those centra, which by their great size are referable to the root of the tail close to the sacrum, bear no chevron mark. These centra are short and wide; their articular ends are both gently concave, and they have a depressed heart-shaped contour. In one of the best preserved the horizontal diameter of the articular end is 100 millims., that of the anterior end being about 90 millims., whilst the vertical diameters of the same ends are 64 millims. and 61 millims., and the length of the centrum measured along its neural surface is 52 millims. At their middle these centra are much contracted. The transverse process stands off from the side of the centrum horizontally just below the line of the neurocentral suture. The arch is low; the spinous process has a strong backward slant; the post-zygapophyses are large and prominent. Other caudal vertebra smaller than those just described differ from them mainly in the presence of chevron marks and in the greater concavity of their articular ends. At least 13, probably about 17 or 18, of the vertebræ in the fore part of the tail had transverse processes; two smaller vertebræ, which by their very small size belonged to near the end of the tail, have a simple cylindroid figure; the processes have disappeared, and the arch is reduced to a tubular hoop (Plate 75, figs. 3, 4).

* Rather less than others.

† The numbers in this line, except the last, are approximate.

The osseous tissue of all the vertebræ is coarse, and the outer surface wants the closeness of texture and the smoothness so observable in Iguanodont bones. This textural difference serves to distinguish very mutilated pieces. It is not peculiar to this particular skeleton, for it is equally apparent in a thoracic vertebra which I dug a quarter of a mile to east of the place where this skeleton was discovered—a distance which makes it almost impossible that it should have ever formed part of it.

Ribs.—Many of these show a double vertebral articulation by head and tubercle. In those referable to the front of the chest, as has been already mentioned, the head of the rib is very swollen and the neck is relatively slender. These parts are wanting in all the disconnected ribs from this part of the chest, but they are fortunately preserved *in situ* upon three vertebræ (Plate 76, figs. 1 and 2). In this region the neck of the rib is short, and it joins the body of the rib in an almost uniform curve. In ribs referable to the middle of the trunk a very decided angle marks the junction of the tubercle and body (Plate 76, fig. 2). In all ribs with double vertebral articulation the neck is slender, and it is so compressed that the long diameter of its cross-section is vertical. Beyond the tubercle the upper border (here become outer) expands so widely as to deserve the term "surface," whilst the lower (here inner) border continues thin. This gives a triquetrous form and great strength to this part of the rib.

Limbs.—The femur (Plate 74) is remarkable for the largeness of its articular extremities and the slenderness of its shaft. Both ends are much damaged by pressure. My description is taken from the right, which is better preserved than the left. The proximal end bears at its inner angle a sessile sub-hemispherical head (*cap.*), external to which and nearly in the same level with it is a well-developed trochanter (*tr.m.*). The inner or posterior trochanter (fig. 2, *tr.i.*), characteristic of dinosauria, is also strongly developed; it is rather nearer to the proximal than the distal end of the bone. The knee condyles are very large; the inner is somewhat the larger of the two; a deep narrow groove separates them in front, and a wider, relatively shallow, depression divides them behind. The length of the right femur is 55·5 centims.; the diameter of the distal end across the condyles is 15·5 centims., and that of the proximal end is 18 centims. The diameters of the caput femoris are 9 centims. and 8·5 centims., and that of the middle of the shaft is 8 centims.

The Tibia (Plate 75, figs. 1, 2), the left one, is much shorter than the femur, being only 35 centims. long. It resembles the femur in the largeness of its joint ends and the slenderness of its shaft. The proximal end, distorted by pressure, shows obscurely a division of the articular surface into two parts answering to the femoral condyles (*cd.*), and a remarkably large præcnemial process (*prc.*). The distal end, flattened, is of the usual dinosaurian pattern, having a longer and narrower outer and a shorter and stouter inner division. When the bone is held vertically so that a line drawn between the proximal condyles is directed forwards, the longer axis of the distal end intersects this, making with it an angle of about 60°. This different direction of the ends makes the shaft appear twisted. Relatively to the large size of the articular ends the shaft

looks singularly short and slender. Its cross-section at the middle is roughly trigonal, and the diameters here are 4·5 centims. and 5·5 centims. Owing to the twist of the shaft, that which below is the inner border becomes as it ascends the broad anterior surface above; and the narrow surface at the upper end lying between the outer condyle and the præcnemial crest as it descends becomes towards the distal end the broad antero-external surface.

Of the *fibula* only a small fragment attached to the outer border of the antero-external surface near the lower end now remains.

Foot.—The only parts of this which can with certainty be identified are two metatarsals (Plate 72, fig. 3). In the great size of their joints and shortness and slenderness of the shaft they repeat [illegible] of the femur and tibia. Their distal end has the usual pulley form. The condyles project strongly towards the sole, and the pits for the attachment of the lateral ligaments are large and deep. The proximal end is most expanded vertically. The length of the best preserved metatarsal bone is about 8·5 centims., the breadth of its distal end is 4·5 centims., the longer diameter of its proximal end about 7·5 centims., and the diameter of the middle of the shaft nearly 2 centims.

Respecting the bones considered unguals by Mr. Fox I cannot speak confidently. Their form is broad, depressed, and blunt.

Dermal armour.—Together with the endoskeletal bones a highly developed dermal armour was found. Mr. Fox told me that when he first laid this bare it formed so continuous a mail that his first impression was that he had exposed the carapace of a huge turtle. It measured 3 feet by 3 feet 3 inches, covered the loins, and it was thinner at its middle near the vertebral column than towards its borders. It is now I fear irreparably damaged and beyond reconstruction. Broken up into countless pieces through hasty and incautious removal from the cliffs, these have in 15 years cracked and fallen into numberless smaller fragments; the attempt to rejoin which would be a hopeless undertaking.

Scutes of three forms are readily distinguished: α, simple flat scutes; β, keeled scutes; and γ, spined scutes. Those of the first kind are most numerous. Since none are now entire their shape and dimensions can only be approximately ascertained. They varied greatly in different situations, since some were certainly more than 26 centims. across, whilst the breadth of others did not exceed 1 centim. Their thickness ranges between 3 centims. and 5 centims. Their deep or inner surface is smooth and their outer surface is studded with scattered tubercles. (Plate 71, fig. 3.)

A smaller number of scutes are keeled. The keel, in some, itself unsculptured, rises within a circle of one or more rows of tubercles which are separated from the margin of the scute by a sunken groove, the inner lip of which is thin and projects. (Plate 70, figs. 3, 4, and Plate 72, fig. 4.) It is probable that into this groove fitted the thin edge of the adjoining scute, a mode of articulation giving flexibility with security against dislocation. The deep surface of these scutes is smooth and sinuous. A few

of the keeled scutes have the deep surface angularly excavated. (Plate 71, fig. 7. Plate 73, figs. 1–4.)

The spined scutes are fewer than either of the other two forms. They are all asymmetrical. Their form is rudely triangular, the shorter are obtuse and the longer acute. (Plate 71, figs. 4–6. Plate 76, fig. 1.)

Their base is very stout, its outline is a rhomboid. The blade projects in one of the most perfect, which, however, wants the tip, to 30 centims. beyond the base. The long diameter of the base of this scute is 21 centims. and the shorter 11 centims. One edge of the blade is relatively straight and the other is incurved. A similar difference in the direction of the borders is seen in the smaller scutes of this kind. When a spined scute is placed upon its base on a flat surface the slant of the blade is seen to be considerable; one surface, which in this position is upper, is nearly plane or sinuous transversely, whilst the other surface is transversely convex. The plane or sinuous surface has its distal moiety deeply furrowed by vascular grooves.

With regard to the distribution of the different forms of scutes we have to guide us: α, Mr. Fox's impressions of the armour as he saw it first before it was disturbed and broken up—he says that from its relations to the other bones he thought it covered the loins as a continuous shield; β, inferences drawn from the scutes themselves; γ, the preservation of a few scutes *in situ* in two regions.

The upper surface of the sacrum is still overlaid by a continuous flat scutal covering ornamented with tubercles, which dot it irregularly without definite grouping (Plate 71, fig. 3). I did not detect in it any marks of joints, and am therefore disposed to regard this as forming part of one large plate, which is certainly in its natural position.

The fortunate recovery of the piece sketched in Plate 73, figs. 1–3, shows that an upper row of carinate angularly excavated scutes covered the neural, and a lower row of similar scutes embraced the hæmal spines of the tail. One of these keeled hollowed scutes, which from its large size was probably situated at the root of the tail, is 21 centims. long, 12·5 centims. high, and the angular excavation of its base is 4 centims. deep (Plate 73, fig. 4). Another from near the end of the tail is only 2·5 centims. long by 5 centims. across, and its keel is quite dwarfed (Plate 75, fig. 5). The upper and the lower row of these keeled scutes did not quite meet, but they left a lateral interval filled by a series of smaller flat scutes (Plate 73, fig. 1). Both forms with the diminution of the bulk of the vertebræ underwent a corresponding reduction, and they became towards the end of the tail small button or buckler-like studs, one of which is shown by Plate 75, fig. 6. Thus the whole tail was sheathed in armour.

It has been already mentioned that the spined scutes are asymmetrical. This alone would make it most unlikely that they formed a median dorsal crest. That they were not so placed in the lumbo-sacral region is demonstrated by the preservation of flat scutes there *in situ*. It is not improbable that the spined scutes and the unexcavated carinate scutes encircled with tubercles were grouped in lateral rows.

As no part of the endoskeleton referable to the scapular region and neck was discovered, it cannot be ascertained how far forwards the dorsal shield reached, neither from the material in its present state can any inference be drawn of the presence of ventral armour. The presence of a hæmal series of tail scutes suggests that the belly as well as the back may have been mailed.

The tissue of the scutes is distinctly bony. The vascular canals in the cortex are large and very numerous (Plate 70, fig. 4), suggestive of a stout epidermal covering.

These remains indicate an animal of low stature whose height at the rump probably did not exceed 3 feet. Its strongly marked bones and their large joints speak of its immense muscular power, whilst the shortness of its limbs and the anchylosis of the lumbar vertebræ welding the loins and the sacrum into a long inflexible rod suggest an absence of the lithesome and agile movements of a terrestrial carnivore, and give probability to its having been a slowly moving vegetable feeder.

As regards its zoological position, its dinosaurian marks—the inner femoral trochanter, the lower end of the tibia, and the forked ribs—are so plain that its reference to this Order cannot be doubted. Its place within the Order is also not uncertain. From the Iguanodont family, as represented by its two best known genera *Hypsilophodon* and *Iguanodon*, *Polacanthus* differs widely in the form and proportions of its limb-bones and vertebræ, and by its very highly developed dermal armour, in comparison with which the scutes of *Hypsilophodon* and *Iguanodon* may without inaccuracy be called flimsy. In its stoutly sheathed, crested tail, and its strong body mail *Polacanthus* repeats two striking features of the Liassic *Scelidosaurus*. The scuted caudal vertebra of *Polacanthus* sketched in Plate 73, fig. 1, presents a resemblance to the figure of a corresponding vertebra accompanying Professor R. Owen's "Monograph on Scelidosaurus," which must strike the most superficial observer.

The trunk armour of *Scelidosaurus* is, however, much less developed than that of *Polacanthus*, the spined scutes of the former, so far as these are known from the types preserved in the British Museum, are smaller than those of *Polacanthus*. The resemblance of *Polacanthus* and *Scelidosaurus* is not restricted to their armour, for massive joints are a feature common to both; the differences of shape and proportion of the limb-bones and vertebræ more than suffice, however, to prove their generic distinctness.

Omosaurus armatus, Owen, of oolitic times, had as large dermal spines as *Polacanthus*, but their shape is very different. The dermal spines of the Jurassic *Stegosaurus* figured by Professor O. C. Marsh, in 'American Journal of Science,' vol. xix., March, 1880, plate x., are apparently not smaller than those of *Polacanthus*, but their form differs from these; the limb-bones of *Stegosaurus* are also more slender and the femur wants the inner trochanter. From the lower chalk *Acanthopholis* (Huxley), which has also a somewhat similar mail, *Polacanthus* differs in the greater development of this protective covering and in its very dissimilar vertebræ.

It is to the Wealden *Hylæosaurus* that *Polacanthus* appears most closely related.

The resemblance of their dermal spines is very close, and their tibiæ are remarkably alike. Indeed, I do not know in any public or private collection any bone which the tibia of *Polacanthus* so nearly resembles as the type tibia of *Hylæosaurus* preserved in the national collection and a tibia which a few years since I brought before the Geological Society and provisionally referred to this dinosaur.* It was obtained from Brixton Bay, the locality which yielded these remains of *Polacanthus*. We may not safely compare the metatarsals of this skeleton with those accredited to *Hylæosaurus*, since the type specimen (No. 2556 Brit. Mus. Cat. figured in the Brit. Foss. Rept., Monog. *Hylæosaurus*, plate xi.) is only conjecturally assigned to it, and may have belonged to a very different animal, for the length and slenderness of these metatarsals are not in harmony with the Hylæosaurian tibia so short and with such expanded articular ends. Neither are the type specimens of sacrum and the disconnected vertebræ in the British Museum available, because their reference to *Hylæosaurus* is also conjectural and still requires confirmation. Such comparison, whatever its value may finally prove, however, shows that the sacrum of *Polacanthus* is much more massive and the thoracic vertebræ are longer and less stout than those reputed Hylæosaurian fossils.

Explanation of Plates.

PLATE 70.

Fig. 1. Posterior view of a vertebra from the front of the chest.†

Fig. 2. Oblique lateral view of the same.

Fig. 3. Fragment of a large keeled scute with grooved margin and submarginal rings of tubercles.
k. Beginning keel.

Fig. 4. Sectional view at *x* in fig. 3. (This and fig. 3 are slightly reduced.)

PLATE 71.

(All the figures in this plate are represented rather less than one-half their natural size.)

Fig. 1. Ventral view of sacrum and anchylosed lumbar vertebræ. The numerals 1–5, *s.*, mark the sacral, and the letters *l.l.l.* the lumbar vertebræ.
ng. Nerve-groove.

* 'Quarterly Journal Geological Society,' vol. xxx., 1874, plate xxxi., figs. 1, 2.

† In all the representations of vertebræ. *c*. Centrum. *ns*. Neural spinous process. *prz*. Præzygapophysis. *pzz*. Postzygapophysis. *d*. Diapophysis. *p*. Parapophysis. *c.c*. Capitulum costæ. *r*. Rib-shaft.

Fig. 2. Cross sectional outline at fracture through the third sacral vertebra.
Fig. 3. Dorsal view of fragment of the large scutal shield resting on the fourth and fifth sacral vertebræ.
Fig. 4. Lateral view of a large dermal spine.
Fig. 5. Basal view of the same.
Fig. 6. Edge view of the same.
Fig. 7. Keeled scute with angularly excavated base.
The border *x* in this and in fig. 4 correspond.

PLATE 72.

Fig. 1. Oblique view of a caudal vertebra.
Fig. 2. Anterior view of the centrum of the same.
Fig. 3. Oblique lateral view of two metatarsal bones.
Fig. 4. Fragment of a large keeled scute with grooved border and submarginal ring of tubercles.
k. Keel.
g. Groove.

PLATE 73.

Fig. 1. Lateral view of a fragment of the tail, sheathed by hæmal and neural keeled scutes, between which are the remains of a lateral series of small peltate scute tubercles. The vertebral centra, *c.c'.*, are overlaid by bundles of ossified tendons, *t.*
Fig. 2. Sectional view of the fractured surface at *c'*.
c. Crushed centrum.
k.k. Hæmal and neural scutes.
Fig. 3. Foreshortened view of scute at *x*. *b.* marks the same border in this and in fig. 1.
Fig. 4. Lateral view of a large keeled scute with angularly excavated base.

PLATE 74.

Fig. 1. Front view of the right femur (nearly $\frac{2}{1}$).
Fig. 2. Back view of same (nearly $\frac{2}{1}$). (This is represented upside down.)
cap. Caput femoris.
tr.m. Trochanter major.
tr.i. Inner trochanter.

PLATE 75.

Fig. 1. Front view of the left tibia (nearly $\frac{2}{1}$).

Fig. 2. Back view of the same (nearly $\frac{2}{3}$).
cd. Condyles.
pr.c. Præcnemial crest.
f. Fragment of fibula.
Fig. 3. Side view of a caudal vertebra, where the transverse process has disappeared, and the spinous process has become dwarfed.
Fig. 4. View of posterior surface of fig. 3.
Fig. 5. Small keeled scute upon a vertebra from near the end of the tail.
Fig. 6. Small buckler-like scute from near end of tail.

PLATE 76.

Fig. 1. Lateral view of a large dermal scute (slightly reduced).
Fig. 2. Fragment of a rib.

XVI. *On the Histology and Physiology of Pepsin-forming Glands.*

By J. N. Langley, *M.A., Fellow of Trinity College, Cambridge.*

Communicated by Dr. M. Foster, *F.R.S.*

Received March 11,—Read March 24, 1881.

[Plates 77, 78.]

Contents.

Page

Rana temporaria.

The structure of the œsophageal and gastric glands.

The œsophageal glands ... 664
The oxyntic glands ... 665
The glands in the junction of the œsophagus and stomach ... 667
The pyloric glands ... 668
The glands in the intermediate zone ... 669
The changes which occur in the œsophageal glands during digestion ... 669
The changes which occur in the œsophageal glands when the animal is fed with sponge ... 671
On some disputed points in the histology of the œsophageal glands ... 672
The changes which occur in the oxyntic glands ... 674
The changes which occur in the pyloric glands and in the necks of the oxyntic glands ... 677
The pepsin-content of the œsophageal and gastric glands ... 678

Bufo variabilis.

Histology of the œsophageal and gastric glands ... 683
Changes which occur in the œsophageal and gastric glands ... 684
Pepsin-content of the œsophageal and gastric glands ... 685

Triton tæniatus.

Histology of the gastric glands ... 686
Changes which occur in the oxyntic glands ... 687
Pepsin-content of the gastric glands ... 689

Triton cristatus ... 690

Coluber natrix.

Histology of gastric glands in hunger and digestion ... 691
Pepsin-content of various regions of the stomach in hunger and digestion ... 693

General Conclusions and Remarks.

Pepsin arises from the cell-granules ... 694
The cells store up zymogen and not pepsin ... 696
The capability of distinguishing zymogen in living glands ... 697
The causes of the morphological changes in any one gland-cell ... 698
Comparison of the effect of digestible and indigestible food ... 702
The causes of the different morphological changes in different gland-cells ... 703
In the cells there is during secretion an increase of substance capable of reducing osmic acid 706

THE following paper contains an account of observations upon *Rana temporaria, Bufo vulgaris, Triton tæniatus, Triton cristatus, and Coluber natrix.* In these animals I have examined the structure of the resting stomach and noted the alterations which occur in it during secretion. I have also estimated the relative amounts of pepsin contained by different portions of the stomach, and the amount of pepsin contained by a definite weight of the gastric mucous membrane in the resting and in the active state. I have further attempted to ascertain whether pepsin exists as such, or in a combined form, in the gland-cells.

I do not propose to give a complete account of the structure of the resting stomach in each animal, although certain points in which my observations differ from or extend those of previous observers I may have to treat somewhat fully.

I shall first describe the individual peculiarities which occur, and shall then discuss them with a view of drawing some general conclusions.

RANA TEMPORARIA.

STRUCTURE OF ŒSOPHAGEAL AND GASTRIC GLANDS.

The œsophageal glands.—These glands have been described by SWIECICKI,* NUSSBAUM,† and PARTSCH.‡ The glands are of the complex tubular type;§ amongst the proper secreting cells are mucous cells, these occur in smallest number in the final dilatations of the ducts. In the ducts ciliated cells are sometimes, though rarely, to be seen. The secretory cells are cylindrical or conical and are smaller than the gastric gland-cells. NUSSBAUM‖ has shown that they contain in the fresh state conspicuous granules; in a teased-out fresh preparation many of these granules are seen floating in the fluid: they are three to five times as large as the granules seen on teasing out similarly the gastric glands; they are even larger than the granules of the pancreas.

The granules have the following reactions:—They dissolve readily in hydrochloric acid 0·4 per cent., less readily in weak alkalies. Bile dissolves them almost instantaneously. Alcohol, varying in strength from 50 per cent. to absolute, dissolves them in part but not entirely; with each granule an undissolved residue is left. I conclude that the solution is real and not simply caused by the extraction of water, since a like effect is not produced by glycerine or saturated solution of sugar. On adding alcohol the granules sometimes run together before the partial solution takes place. Thus, in one

* SWIECICKI, PFLÜGER'S Archiv., Bd. xiii., s. 444, 1876.

† NUSSBAUM, MAX SCHULTZE'S Arch., Bd. xiii., 1877.

‡ PARTSCH, MAX SCHULTZE'S Arch., Bd. xiv., s. 179, 1877.

§ I apply the term "simple tubular" to such glands as consist of one tube; when several tubes are given off by one duct, I call the glands "compound" tubular; when the tube or tubes arising from a duct divide, I call the gland a "complex" tubular gland. KLEIN describes the œsophageal glands as acinous glands (STRICKER'S 'Handbook,' vol. i., p. 538).

‖ *Op. cit.*, s. 748.

instance, I watched three granules lying close together; first one ran into its neighbour, then this into the remaining granule, the whole forming one large granule; in it several brighter spots appeared; later, the greater part suddenly vanished leaving four or five rather bright particles arranged so as to produce the appearance of a fragment of a small-meshed network. I have little doubt that the apparent network seen in the cells in alcohol specimens has its origin from these residual particles.

I may mention that the zymogen granules of the pancreas behave in a similar manner with alcohol, so that in alcohol specimens the granules of the inner zone are only the representatives of the actual zymogen granules.

The granules are not obviously affected by irrigation with a 5 per cent. solution of ammonium chromate or bichromate; they disappear however from the gland-cells when a piece of the œsophagus is left for one or two days in either of these fluids, the nuclei then show distinctly a network or a tangle of fibres.

According to HEIDENHAIN* the gland cells of the pancreas after a two to three days' stay in 5 per cent. ammonium chromate show a marked striation in their outer portions. In similarly prepared specimens of the œsophageal glands I have not been able to observe a similar structure, although in osmic acid specimens the outer zone not infrequently has a faint striation.

It was shown by NUSSBAUM† that the œsophageal gland granules are preserved by osmic acid. In treating glands with this reagent I usually use the following method. The tissue is placed in a 1 per cent. solution for twenty-four hours, removed to 50 per cent. alcohol for fifteen minutes, and then transferred to 75 per cent. alcohol. Sections are cut on the following day. In sections so prepared the granules are stained not very deeply and have a yellow-brown tint. The sections as a whole are less stained than similarly prepared sections of the stomach. The tint of staining of the gastric gland granules tends to be brown-black rather than yellow-brown.

SEWALL and myself‡ found that the œsophageal gland-cells occasionally showed clumps of highly refractive granules in their peripheral portions. To distinguish these from the proper granules of the cells, we called them "border" granules. Further investigation has shown me that these are really fat globules. Their position is very constant and they give a striking appearance to osmic acid specimens. Quite similar fat globules are occasionally to be seen in the pancreatic gland-cells. They occur also in the gastric gland cells of the Frog, Toad, and Newt (see Plate 78, fig. 7), although their arrangement is not quite so regular. In a subsequent paper I hope to discuss the causes which influence the appearance of fat globules in the above and in other secretory gland-cells.

The oxyntic glands.—I propose to use the term oxyntic glands (ὀξύνειν—to make

* HEIDENHAIN, PFLÜGER'S Archiv., Bd. x., s. 561, 1875.

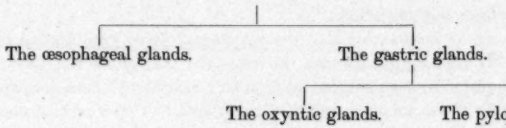
† *Op. cit.*

‡ LANGLEY and SEWALL, Proc. Roy. Soc., Oct., 1879, p. 383; Jour. of Physiol., vol. ii., p. 283, 1879.

sour, to acidulate) for those glands in the stomach which are differently called by different observers "fundus," "peptic," or "rennet" glands. It is only after great hesitation that I venture to employ a new term, but without a new term I find myself reduced to circumlocution or inaccuracy. That the present nomenclature is unsatisfactory scarcely needs to be pointed out. In the Rat there are no glands in the fundus of the stomach; in the Rabbit the glands of the fundus proper differ in some important points from those of the greater curvature, yet both are called fundus glands. The terms peptic and rennet glands are inappropriate, since the pyloric glands also secrete the peptic and rennet ferments. The terms "simple" and "compound" glands suggested by EBSTEIN* are applicable only to the gastric glands of Mammals, for it is only in Mammals that compound glands, *i.e.*, glands possessing both border- and chief-cells, occur.

The one characteristic point of the "fundus," "peptic," or "rennet" glands in all animals is the secretion of an acid fluid. This characteristic is suggested by the word "oxyntic."

In the Frog, then, the glands which produce the secretion active in gastric digestion are



The oxyntic glands have been most fully described by PARTSCH.† The epithelium on the surface of the mucous membrane and that in the mouths of the glands consists of long cylindrical cells, which in their outer portions contain mucigen. Each cell is prolonged into a fine process. In the necks of the glands are found, in the upper portion, nearly cubical cells, in the lower portion two or three very marked mucous cells. In the body of the gland are the proper secretory cells; they are rather irregular in form, but have a tendency to be ellipsoidal. When the cells are partially isolated after treatment with neutral ammonium chromate 5 per cent. they are frequently seen to possess a short prolongation corresponding to the process of a mucous cell of the surface. The cells on the opposite sides of a gland-tube are usually so arranged that the nucleus of a cell on one side faces the junction line of two cells on the other. (See Plate 77, fig. 10.)

* EBSTEIN, MAX SCHULTZE'S Archiv., Bd. vi., s. 538, 1870. The words "simple" and "compound" are so commonly used to describe the form of glands, that it would probably lead to some confusion to use them for glands consisting respectively of one or of two kinds of cells.

† *Op. cit.* The earliest account I have met with is that of HEIDENHAIN, MAX SCHULTZE'S Archiv., Bd. vi., s. 394, 1870.

When the muscular coat is removed from a fresh stomach and the mucous membrane pinned out with the muscularis mucosæ uppermost, the glands do not as a rule show distinct granules, but present a ground-glass appearance. When the mucous membrane is thin and the light good, small granules of scarcely greater refractive power than the cell-substance in which they lie can be seen with Zeiss' obj. D, oc. 2. The cells are not filled with obvious granules, as are the œsophageal gland-cells, but, on the other hand, they are not clear and transparent like the cells of the pyloric region.

The small granules come out distinctly on teasing a portion of the mucous membrane in salt solution* (0·6 per cent.) or, better, in water. The small size of the granules and their slight refractive power make it difficult to observe the action of reagents on the individual granules. The reagents, however, mentioned above as dissolving the œsophageal gland granules—viz: bile, dilute acids, and alkalies—soon make the cells transparent, leaving in them little or no trace of the granules which previously crowded them. It would appear, then, that the granules of the œsophageal and oxyntic glands resemble one another in certain general characteristics.

On treatment with neutral ammonium chromate (5 per cent.) the nuclei of the oxyntic gland-cells show a network like, but less distinct, than that described by Klein† in the similar cells of the Newt.

The glands in the junction of the œsophagus and stomach.—The characteristic œsophageal and oxyntic glands just described are separated from one another by intermediate forms. The last two or three millimetres of the œsophagus and the first one or two millimetres of the stomach contain many transition-forms between the two. Partsch has mentioned that near the stomach the œsophageal glands lose their complex tubular form and pass into the simple tubular gastric glands. They do not, however, regularly and in succession become more and more simple; there are many irregularities. Here and there may occur what is little more than a depression of the surface epithelium, or there may be a return to the complex gland. The mucous membrane in this intermediate region is thinner than that either above or below it. Swiecicki, from the examination of hardened specimens, described the œsophageal glands as stretching into the cardia. What we, in fact, see when the fresh mucous membrane is stretched out is that in the intermediate region the glands are fairly equally scattered throughout, and are not arranged in packets with intervening spaces as in the œsophagus, but that, nevertheless, the first part retains the characteristic œsophageal gland granules. When this intermediate region is treated with osmic acid, and subsequently with alcohol, we find that the first simple tubular glands which occur have rather large yellow-brown-stained œsophageal granules (cp. Plate 77, figs. 1, 2, and 3), whilst farther backwards these granules begin to be replaced in some of the gland cells by the small brown-black-stained oxyntic-cell granules. We have, then,

* Salt solution makes the glands at first more cloudy; then the cloudiness disappears and the granules become obvious.

† Klein, Quar. Jour. Mic. Soc., vol. xviii. (new ser.), July, 1878, p. 315, *et seq.*

glands with some cells resembling in the main the œsophageal, and others resembling the oxyntic gland cells. Farther from the cardia the œsophageal granules diminish still more in size, so that they are scarcely or not at all larger than the oxyntic-cell granules. In some cases the granules can be referred without much difficulty to one type or the other; in other cases they cannot. Occasionally one or more cells with large "œsophageal" granules occur in oxyntic glands at some distance from the cardia. The granules in the anterior oxyntic glands are, as a rule, rather larger than those in the posterior.

The pyloric glands.—The pyloric region of the stomach forms about one-fifth to one-fourth of the length of the whole stomach; it is recognised under the microscope by the transparency of its cells. PARTSCH* compared the pyloric gland to the mouth and neck of an oxyntic gland. The comparison is, I think, just. The cylindrical cells of the surface of the pyloric mucous membrane become shorter and shorter, and pass without any break into the sub-cubical cells which form the greater part of the glands; below these are usually, though not always, one, two, or more distinct mucous cells. When the glands between the oxyntic and pyloric glands are examined it is seen that the mucous cells at the lower part of the pyloric glands correspond to the mucous cells at the lower part of the neck of the oxyntic glands.

If the stomach of a hungry Frog is hardened in alcohol, and sections cut, it is seen that the sub-cubical cells of the oxyntic and pyloric glands closely resemble one another, and further that both closely resemble the cylindrical cells.

The outer portion of the cylindrical cell consists, as we know, mainly of mucigen. In alcohol specimens this mucigen portion is transparent and sharply marked off from the protoplasm which forms the rest of the cell. Now, in alcohol specimens the outer portion of the sub-cubical cells, both in the oxyntic and pyloric glands, is similarly marked off; that is, the outer portion of the sub-cubical cells also consists mainly of mucigen. The two kinds of cells, then, resemble one another in having a protoplasmic inner portion and a mucigenous outer portion. They differ somewhat in shape: the one is usually a four-sided wedge, tapering to a fine point; the other more approaches a cube in shape, with a process from the base which bends round and overlaps the cell next below it. If the process were straightened and the cell elongated a little we should have a "cylindrical" cell. The position of the processes of the sub-cubical cells is exactly similar to the position of the processes of the mucous cells; probably, indeed, these cells differ from one another chiefly in the extent to which they form mucigen. In the above description I have added but little to the account given by HEIDENHAIN, PARTSCH, and NUSSBAUM.

Osmic acid specimens prepared as above described (p. 665) do not show the resemblance of the cylindrical and sub-cubical cells equally clearly. The former have the mucigen border fairly well marked, but the latter are much more equally stained throughout: the mucigen border is only shown by a somewhat lighter yellow-brown

* *Loc. cit.*

coloration.* Neither show any granules. The distinction comes out, however, on keeping the specimens in glycerine (Plate 77, fig. 2), partly by the protoplasmic portion becoming darker, and partly, I think, by the mucigen portion becoming lighter. The distinction is also clear if the osmic-hardened stomach is left in alcohol a week or more before sections are made. In the sub-cubical cells the nucleus is placed in the outer portion of the cell and takes up nearly the whole of its transverse diameter.

The glands in the intermediate zone.—In the intermediate zone between the oxyntic and pyloric glands, the glands become shorter, the ellipsoidal cells of the oxyntic glands become fewer, and some of them are replaced by cells similarly shaped but containing few or no oxyntic gland granules; amongst these glands are found simple pyloric glands, which increase in number towards the pylorus until they form the sole constituent. Close to the intestine the glands are very irregular in form, and become more and more simple depressions of the surface epithelium.

The Changes which occur in the Œsophageal Glands during Digestion.

Sewall and myself† have previously given some account of the most striking event of secretory activity, viz.: the using up of the cell granules. We found that in the normal hungry Frog the cells were granular throughout and that very soon after feeding the animal the granules began to disappear, and continued to disappear until about the sixth hour; at some later period which we left undetermined the granules began to increase, and increased steadily until the cells were again granular throughout.

To this account I would make one or two additions.

When a Frog is fed, the œsophageal glands near the stomach show greater signs of secretory activity than glands more remote. This is the case, at any rate, when only a moderate amount of food is given. The glands which are nearest the stomach are the first to show a clear zone, then those just above, and so on to the beginning of the œsophagus.

Generally speaking, the smaller the amount of food given the more is the obvious effect confined to the œsophageal glands that are near the stomach, and within certain limits the more digestible the food and the greater its amount the more simultaneous is the change taking place in the glands.

Hence, in comparing the state of the glands at different times after food has been given, it is important to take a strip down the whole length of the œsophagus; and in comparing the amount of pepsin in different stages of digestion it is important to take pieces of the œsophagus from a corresponding region.

As the outer zone increases the granules in the inner zone become smaller, the

* The mucous cells are also fairly equally stained in specimens treated with osmic acid only. This probably explains how it is that Bleyer (quoted by Partsch) failed to observe mucous cells in osmic acid specimens.

† *Op. cit.*

diminution in the size of the granules is very marked in cells in which the outer zone takes up the larger part of the cell. After great activity there is also an obvious diminution in the size of the cells, and although it is difficult to be certain of the changes which take place in the first stages of activity, I have little doubt that the diminution in the size of the cell begins with the development of the clear outer zone.

In specimens treated with osmic acid there is another effect of activity to be seen, viz.: the cell-substance stains more deeply than during rest. In the œsophageal glands the tint of staining is not in so constant a relation to the amount of secretion produced as in some other glands, the extent to which the tint deepens seems to vary as the stimulation is produced by digestible food or by mechanical stimulation. These and some other apparent causes of variation have not been closely determined, but the main fact of an increase in the depth of staining has always been obvious.

It was stated by SEWALL and myself* that absolute alcohol added to the fresh teased-out gland altered the normal appearances; and I have said above that the granules are in part dissolved. Nevertheless, alcohol specimens of œsophageal glands taken during digestion show the two zones in the gland-cells; the non-granular zone stains with carmine, and thus specimens can be obtained (Plate 77, fig. 8), which, except for the smaller size of the cells, closely resemble similarly prepared specimens of the pancreas. It is almost unnecessary to remark that the "granules" of the inner zone in alcohol specimens are not the granules present in the fresh gland. With ZEISS', oc. 2, obj. E or F, the granular zone appears as a fine network.

Nothing very definite can be said as to the time after feeding at which the changes in the œsophageal glands occur. When Frogs are taken as nearly as possible alike, and they are treated in the same way, then the results obtained correspond very closely, but when such results are compared with those obtained from Frogs at a different season of the year, with those obtained from Frogs which are older or younger, more or less healthy, or when different amounts of food are given, then considerable divergences occur.

The changes occurring are in each case of the same nature, but the extent to which these changes take place varies largely. Hence any estimation made of the time taken for the first appearance of a clear zone, for its maximum development, and so on, can only be approximate.

During the first hour or hour and a half after feeding, no distinct change is to be seen. After this period a diminution in the number of the granules in the outer half of the cell becomes obvious. Usually this is first seen in the glands close to the stomach. The disappearance of granules in the outer portion of the cell goes on, so that a clear zone is formed. The clear zone steadily increases until the sixth to twelfth hour, or even later, the time varying with the state of the animal and the amount of food given. The glands then begin to become more granular; the time of complete recovery varies enormously; in some cases the glands are throughout granular

* *Op. cit.*, p. 283.

in twenty-four hours from the time of feeding the animal, in others they do not become so for several days. It will be noticed that the granules begin to increase before the food has left the stomach.

The Changes which occur in the Œsophageal Glands when the Animal is fed with Sponge.

Hitherto I have spoken of the effect of feeding with worm, *i.e.*, with a readily digestible substance; having in mind Heidenhain's experiments on the isolated fundus in Dogs I was anxious to see what would be the effect on the œsophageal glands of mechanical stimulation of the gastric mucous membrane. To this end a number of Frogs were fed with sponge.

If the piece of sponge is small so that it can pass the pylorus a slight effect only is produced: usually a thinning of granules in outer portion of the cells of the anterior œsophageal glands, and a small zone in the posterior œsophageal gland-cells.

When the piece of sponge is too large to pass the pylorus, it serves as a continual stimulus to the œsophageal glands. The extent of the change produced is within certain limits the greater the larger the piece of sponge; it varies, too, widely with the condition of the Frog. The changes produced are like those produced by feeding with worm, but go on very much more slowly. The first distinct thinning of granules is usually not seen for three or four hours, and may not be obvious till even later. The glands near the stomach are first affected. The disappearance of granules then goes steadily on.

It will be remembered that in the worm-fed animal the granules begin to increase in six to twelve hours. After feeding with sponge no such increase occurs until at any rate some days.

The extent to which the disappearance of granules proceeds varies in different cases: in many cases two days after feeding* with a rather large piece of sponge, occasionally in a less time, scarcely any granules are left; and in some glands not a granule is to be seen (Plate 77, fig. 6 (*a*), 6 (*b*)).

When the clear zone is largely developed there are usually to be seen in it, often in rows, fine granules much smaller than those which form the remains of the granular zone (see Plate 77, fig. 6 (*b*)).

At this stage the diminution, both in the size of the cells and in the size of the granules, is very striking. Moreover a very characteristic appearance is imparted to many of the glands by the large size of the lumen. The diameter of the lumen varies considerably in neighbouring glands; in some it is more than half the length of the cells; when granules remain they form a kind of ragged fringe to it (Plate 77, fig. 5).

* Nussbaum (*op. cit.*, s. 749) made some observations upon the *direct* stimulation of the œsophageal mucous membrane, the cardia being ligatured before the animal was fed with sponge. He found under such circumstances that the granules entirely disappeared from the cells in three to five hours. Sewall and myself (*op. cit.*, p. 285) were unable to observe any such rapid action.

If at any time the sponge is vomited by the Frog, or if the sponge be removed, the glands begin at once, or almost at once, to return to the normal state; and in one to two days the lumen is no longer visible, and the cells are crowded with large granules.

We know from the experiments of HEIDENHAIN that in Mammals the mechanical stimulation of one part of the stomach causes only a temporary secretion from the glands of other parts. In the Frog the case is different, the mechanical stimulation of the stomach causes a considerable secretion from the œsophageal glands—a secretion which lasts several days at least.

On some Disputed Points in the Histology of the Œsophageal Glands.

In the account of the changes in the œsophageal glands given by SEWALL and myself there were two points which clashed with the earlier observations which NUSSBAUM made on osmic acid specimens. He described the gland-cells of the normal hungry Frog as having a large clear zone, and found that on feeding the animal the granules increased so that in three to five hours a clear zone was no longer to be seen.

GRÜTZNER's* results suggest an explanation of the divergence between the account of NUSSBAUM and that of SEWALL and myself. He finds that in the normally hungry Frog the œsophageal glands are granular throughout and diminish in granularity during digestion; but finds also that if a Frog is kept longer than usual without food a clear zone is then formed in the œsophageal glands and that on feeding there is at first an increase of granules. Thus according to GRÜTZNER, NUSSBAUM's results would represent what occurs in a pathological and not what occurs in a normal condition.

During the last year and a half I have made a considerable number of observations with the view of determining the points at issue. As regards the state of the glands in the normal hungry Frog I have seen no reason to alter my first-formed opinion.

In some Frogs a large clear zone in the œsophageal glands does occur, namely, in those in which there are signs of general inflammation. In nearly all cases in which I have found a marked clear zone in the gland-cells of a hungry Frog, the animal had some mark or other of an ill state of health. I have frequently selected lively, active Frogs, and sluggish, unhealthy ones from a batch brought to the laboratory, and in a few days examined the œsophageal glands. The gland-cells in the former were granular throughout; those of the latter had almost always a clear zone.

There is one condition in which I have found apparently healthy summer Frogs, kept without food for three or four days, show a clear zone in the œsophageal glands, viz.: when they have remaining in the stomach some piece of stick or leaf, or other undigested substance; such undigested material causes a continuous secretion from the œsophageal glands (see action of sponge, p. 671).

* GRÜTZNER, PFLÜGER's Arch., Bd. xx., s. 395, 1879.

During prolonged fasting a diminution of granules occurs. The amount of diminution varies in different Frogs, and varies with the time of year: it is less in winter than in summer. In most cases the granules only become fewer at the outer borders of the cells; in others a clear zone is formed. Thinning of granules, however, rather than the formation of a clear zone, seems to me to be the normal effect; for in all the *perfectly healthy* December Frogs I have examined, the granules, though fewer at the outer part of the cells, still extended to the periphery. Some diminution in the size of the cells also takes place.

I conclude, then, that in a healthy Frog during the winter months fasting does not necessarily produce a distinct non-granular zone in the œsophageal glands. In other months when the tissue change is more rapid, fasting may produce a non-granular zone, but I think the most usual and effective cause of such a zone is some alteration in the general condition of the body, independent of fasting, by which the normal equilibrium in the gland-cells is disturbed.

My experiments have only been made on *Rana temporaria;* in other species of Frogs, fasting may have a greater effect, just as the effect is greater in *Triton tæniatus* than in *Triton cristatus.*

It remains to consider how far an increase of granules takes place immediately after feeding. NUSSBAUM considered that the granules always increase in the first hours of digestion, but it is to be remembered that his observations were made on Frogs in which the œsophageal glands had a non-granular zone before feeding. GRÜTZNER only found an increase of granules in the first hours of digestion when a clear zone had been previously developed in the œsophageal glands by long fasting. In both cases, then, the increase is only described as taking place in glands having a non-granular zone to start with.

The method of experimenting contains an obvious source of error. We do not know with certainty what is the state of the glands before feeding, and therefore cannot say positively whether an increase or decrease of granules takes place. My own observations lead me to conclude that in a normal hungry Frog no increase of granules takes place in the first hours of digestion, and that in abnormal Frogs, *i.e.*, in those having already zones in the œsophageal glands, an increase may or may not take place. I have seen only one instance of apparent distinct increase, and I am by no means certain that it was not simply apparent.

I am not prepared to deny that a *slight* increase may not take place in all cases, for I think our present methods do not allow us to detect slight differences in the amount of granules contained by gland-cells. Further, I am strongly of opinion that a formation of granules goes on during the whole digestive period, and I can readily conceive that under certain circumstances the formative might overbalance the excretory processes in the first hours, as they certainly do in the last hours of digestion.

The Changes which occur in the Oxyntic Glands.

Partsch* is the only observer who has described any alteration in the gastric glands during activity. His observations were made upon alcohol-hardened specimens. According to him, when a Frog is fed the cells of the gastric glands increase in size for twelve to eighteen hours, and subsequently to this diminish, so that in about twenty-four hours they have returned to their normal condition. This account I cannot in any point confirm.

I will first briefly state what the changes are which I find do occur. Some are like those which occur in the œsophageal glands, viz.: the cells become smaller; the granules become smaller and less distinct; the lumina become apparent; there is an increase in the cell-protoplasm *s.* substance staining with osmic acid. These are the only changes which normally occur in those glands which immediately follow the intermediate region between the œsophagus and stomach. The number of these glands is not constant; within certain limits the larger the meal the fewer glands there are which do not undergo the changes to be presently described.

In the remainder of the oxyntic glands, forming a considerable majority, there are other additional changes which strongly contrast with those which take place in the œsophageal glands; we have seen that in the œsophageal glands the granules disappear from the outer portion of the cells during activity, *in the majority of the oxyntic glands, on the contrary, the granules disappear from the* inner *portion of the cells during activity.* The other and less important differences which exist will be considered in the course of the following description of the glands.

Examination of fresh specimens.—Several of the changes mentioned above cannot be satisfactorily seen in fresh specimens of the oxyntic glands; in the pinned-out mucous membrane it will be remembered that the glands do not show distinct granules, so that any alteration in their number and size cannot be well observed. The active glands are, however, much more transparent than the resting; this is in part no doubt due to the cells having become smaller and the membrane consequently thinner; but it is also due to a diminution in number and size of the granules, for a distinct difference in granularity is seen in the two specimens on adding salt solution 0·6 per cent.

During strong activity the oxyntic glands near the pyloric region may become almost as transparent as the pyloric glands; nevertheless, they can still be fairly readily distinguished; they are more refractive than the pyloric glands, and with a not too bright light have a very faint yellowish tinge. The oxyntic and pyloric glands of the Newt and Snake show in similar circumstances similar differences.

The increase in the size of the lumen can be seen; it is usually most obvious in the gastric glands near the pyloric region. It is only at the height of digestion after a

* *Op. cit.*, s. 193.

very heavy meal, or in certain abnormal circumstances, that the lumen becomes large and conspicuous.

The disappearance of granules from the inner portion of the cells is little or not at all marked in the fresh specimen. No distinct zones, such as occur in the œsophageal glands, are formed.

Examination of specimens treated with osmic acid.—In order to make out more in detail the changes which take place, the stomach is pinned out in osmic acid for twenty-four hours and then put into alcohol. The best preparations are obtained when the tissue is left in alcohol for several days before sections are made, so that the cells are stained of a black instead of a yellow-brown tint.

In the resting state the glands show no lumina, the cells are crowded with fine granules, and through the granules the nuclei appear as clearer spots, the cell outlines being very faintly marked (Plate 77, fig. 9); the cell-substance is almost unstained.

It will be convenient to consider first the changes which take place in a healthy summer Frog fed with a worm small enough for the stomach to have emptied itself completely in twenty-four hours.

In one to two hours after feeding, the lumina begin to be obvious, and the granules to disappear from the inner borders of the cells. This causes the glands to assume a very characteristic appearance. When examined with obj. C or D (Zeiss) a clearer line is seen to run down the middle of each gland; it is usually of a zig-zag or corkscrew form; the form being naturally dependent upon the arrangement of the cells in the gland-tube (Plate 77, fig. 10).

Up to about the fifth hour these changes become more and more pronounced, and at the same time the cells and the remaining granules they contain become distinctly smaller, and the cell-substance stains more deeply.

It is noteworthy that the granules do not disappear simply from that part of the cell which immediately borders the lumen, but to some extent also at the sides of the cells where they are in contact with one another (see Plate 77, fig. 11). The disappearance of granules at the sides of the cells does not extend to the basement membrane.

At the period of maximum change the nucleus is much larger compared with the cell-substance than it is during rest; it is still surrounded by finely granular protoplasm and is sometimes placed near the outer border of the cell, in this differing from the nuclei of gland-cells hitherto observed. Very frequently the granules appear to be most numerous in the cell-substance immediately on the inner side of the nuclei.

The return to the normal appearances begins about the fifth hour; so that during the greater part of the digestive period the formative processes go on whilst the excretory are still active. In twenty-four hours the glands have nearly or altogether returned to the hungry condition.

The above I regard as the normal round of changes in the oxyntic glands of the Frog during digestion.

In the oxyntic glands, however, as in the œsophageal, the times and extent of the changes vary enormously with the amount of food given and the general condition of the Frog.

If a Frog is fed with several worms so that the stomach is much distended with digestible food, the changes are greater and persist for a much longer time. The diminution in the size of the gland-cells makes itself obvious in a diminution in the length of the glands. In twenty-four hours the glands instead of having returned to the hungry state are still small and consist of somewhat small cells with a more or less distinct inner non-granular border; the lumina are frequently large. These points will be seen in Plate 77, fig. 11, taken from a Frog twenty-four hours after feeding with four worms. In such specimens we are better able to observe the increase in size of the lumen. Generally speaking, as the cells become smaller the lumen becomes larger, but we have to reckon not only with the size of the cells but with the pressure of the surrounding tissue. It is probably due to local variation in this respect that the glands in different regions of the same stomach have lumina of very different size. Usually the lumina are most conspicuous at the base of the glands.

In Frogs to which an excess of food has been given, the non-granular inner zone is usually most obvious about the eighteenth or twentieth hour after feeding. The cells then have increased and are still increasing in size; the greater clearness with which the non-granular zone can be seen is then probably due to the nett increase in the cell-granules taking place more slowly than the increase in the cell-protoplasm. A slower nett increase of cell-granules might clearly result either from granules being used up more quickly or from granules being formed more slowly.

The effect of fasting is not very great in winter Frogs which are subjected to the ordinary winter temperature. The glands and their cells become somewhat smaller, but the granules do not ordinarily disappear from the inner part of the cells.

But in Frogs at other times of the year, or winter Frogs which are kept in the warm, fasting produces a fairly marked secretory appearance; the glands are small, the cells of the posterior gastric region have a small non-granular inner portion, and the lumen is to be seen. When Frogs in this condition are fed the ordinary secretory changes set in, except that in twenty-four hours the glands have not or have scarcely recovered their initial condition. They still differ considerably from normal glands.

We have seen that in the œsophageal glands the maximum change is produced by feeding the Frog with sponge. It is not so with the oxyntic glands; in these feeding with readily digestible substance, as worm, is the most effective means of producing a change. When the oxyntic glands are examined twenty-four to forty-eight hours after feeding the Frog with sponge, no very great divergence from the normal hungry state is to be observed. The gland-cells are smaller, but there is very little sign of a non-granular inner zone, and the lumen is seldom obvious. The amount of acid contained by the sponge shows that the secretory processes have been going on.

If however a Frog that has been fed with sponge two days previously be fed with

worm, very marked signs of secretory activity result. The chief feature is the diminution in the size of the gland-cells; the nuclei are usually oval, frequently lie close to the basement membrane, and are surrounded by only a small amount of darkly staining cell substance (see Plate 78, fig. 1). It is to be noted that this gland is much more highly magnified than the rest). The recovery, too, is long delayed, and an inner non-granular zone is visible in the posterior oxyntic glands for some time after the cells have begun to increase in size. The amount of change produced here—and in the case of feeding fasting summer Frogs—*seems* to be greater than that produced by a like treatment of normal hungry Frogs. I say the change *seems* to be greater, since it is hazardous to institute a comparison between the amount of change produced in glands which start secreting in different states. Assuming, however, that the comparison is just, a not unlikely conclusion is suggested, viz. :. that the formative processes require certain elaborated material, and that during the stimulation of the glands of the fasting Frog with sponge, the elaborated material is largely used up, so that the rapid waste brought by the presence of digestible food cannot be made good so quickly as normally.

It is worth remark that the oxyntic and œsophageal glands do not necessarily show a parallelism in the amount of change they respectively undergo in digestion. Under special circumstances the one or the other may be most affected.

Lastly, we have to consider whether on feeding a Frog there is a preliminary increase in the size of the oxyntic cells or in the number or size of the granules contained by them, before the decrease sets in. From what I have already said, it will be seen that, as far as my observations go, the preliminary increase, if it takes place, can only last a short time. Even of such brief increase I have seen no instance. I am not, however, inclined to deny that it might under certain circumstances take place.

Changes occurring in the Pyloric Glands and in the Necks of the Oxyntic Glands during Digestion.

The changes are of a like nature in both, but usually more strongly marked in the former, and, as might be expected, the changes do not run a parallel course in the two portions of the stomach. My observations have been made upon osmic acid specimens. Of the early stages of digestion I cannot speak with any confidence. The mucigen border in all cells frequently appears to be larger and to bulge more at its free surface; this I am inclined to attribute rather to a swelling of some constituent of the outer part of the cells than to an increase by metabolism. However this may be, there is, at the height of digestion with a heavy meal, a very considerable diminution in the amount of mucigen in all the cells.* The cylinder cells of the surface are affected in a similar manner although to a less degree. All the cells, too, are smaller. The inner portion of the sub-cubical cells in osmic acid specimens instead

* Sewall and myself found similar changes take place in the mucous cells of the œsophagus and of the œsophageal glands during digestion. (Journal of Physiol., vol. ii, p. 284, 1879.)

of appearing homogeneous, shows a network with inter-fibrillar substance. The inter-fibrillar substance takes the form of granules. In sections cut the day after the tissue has been placed in alcohol, the granules are only slightly stained, but if the tissue be left in alcohol for several weeks before sections are made, the granules then stain deeply and are very distinct (Plate 77, fig. 4). The network with its contained granules makes its first appearance at the junction of the protoplasmic and mucigenous portions of the cells (see Plate 78, fig. 1), and as secretion goes on encroaches more and more on the mucigen. A similar network is seen during digestion in the cylindrical cells of the surface of the mucous membrane, but it is usually confined to the junction of the protoplasm of the cell with the mucigen.

The changes which take place in these cells differ somewhat, but not essentially, from the changes which take place during secretion in the mucous salivary glands of Mammals. The protoplasm around the nucleus and the protoplasmic network throughout the cell grow; the growth is more rapid in the peripheral protoplasm. It is not quite clear, however, why the inter-fibrillar substance increases in power of staining with osmic acid.

We have seen that during fasting the granules in the pepsin-forming cells slowly diminish, in the distinct mucin-forming cells the amount of mucigen goes on increasing for some time after the granules of the pepsin-forming cells have begun to decrease. The maximum amount of mucigen is contained by the pyloric and similar gland-cells after a moderately prolonged fast. The minimum amount of mucigen is contained by these cells twelve to eighteen hours after a heavy meal; it is then only with difficulty that the mucous can be distinguished from the sub-cubical cells.

Similar changes to those above described are seen in the stomach of a Frog one to two days after feeding with sponge; the main difference is that the tint of staining is usually lighter than after feeding with worm or other digestible substance.

It is worth remark that although the mucous cells of the necks of the glands in alcohol specimens closely resemble the mucous cells of such glands as the sub-maxillary of the Dog, yet in fresh specimens they differ in one important respect. In the sub-maxillary gland of the Dog the cells are in life densely crowded with distinct granules, whilst the mucous cells of the stomach and, generally speaking, of the alimentary canal are transparent and show no trace of granules.

The Pepsin-content of the Œsophageal and Gastric Glands.

Swiecicki concluded from his observations that the œsophageal glands are at any rate the chief source of pepsin, and that the gastric glands produce little, perhaps indeed none.*

* He says: "Alle diese Thatsachen sprechen hiernach dafür dass bei den Fröschen die Pepsinbildung vorzugsweise, ja vielleicht nur allein in dem Œsophagus von statten geht, während der die Belegzellen führende Magen die Saüre bildet" (s. 452).

NUSSBAUM suggested that the pepsin arose from the œsophageal gland granules. He found a correspondence between the number of granules in the gland-cells and the amount of ferment contained by the glands.

SEWALL and myself, whilst differing from NUSSBAUM as to the times of increase and decrease of granules, came nevertheless to the conclusion that the granules were connected with the formation of ferment. To this conclusion we came partly on general grounds, partly by comparing GRÜTZNER'S results on the times of increase and decrease of pepsin with our own on the times of increase and decrease of granules.

I have made some experiments to determine this point. Since the œsophageal glands are affected during digestion to such different degrees in different Frogs, I have paid no especial attention to the alteration in pepsin-content of the œsophagus which occurs in successive hours after feeding.

I proceeded in the following manner: hungry Frogs and Frogs fed with worm or sponge were taken, and (1) the granularity of the gland-cells observed; (2) the relative amount of pepsin contained by equal weights of dried œsophagus estimated.

The Frogs were taken as much as possible alike in size and general condition. Part of the œsophagus was observed fresh, part after treatment with osmic acid and alcohol—the rest of the mucous membrane of the œsophagus was dried and a definite quantity weighed out, this was treated with HCl. 0·2 per cent.—3 cub. centims. for 0·01 grm.—for twenty-four hours. The filtrate from this was tested for pepsin in the ordinary manner by GRÜTZNER'S colorimetric method. Only those specimens which were analysed for pepsin on the same day were compared.

In some cases of slight difference in amount of granules the results were not constant, this I attribute to the imperfection of the method; equal weights of dried œsophagus do not necessarily contain equal weights of secretory gland-cells.

In all cases where there was a marked difference in the amount of granules there was also a marked difference in the amount of pepsin.

Hence, then, the greater the amount of granules the greater is the amount of pepsin to be obtained from the glands. I think we can fairly conclude that the granules give rise to the ferment.

We can now consider the question, Do the oxyntic glands form pepsin? I have suggested above that SWIECICKI was inclined to attribute the pepsin found in the gastric mucous membrane to an absorption of the pepsin secreted by the œsophageal glands. In this I cannot agree with him. The amount of ferment found in the mucous membrane of the stomach is far too great to allow of any explanation except that it is formed by the gastric glands. It is true that when *equal weights* of the mucous membrane of the œsophagus and stomach are taken and their relative pepsin-content compared, the amount is found to be considerably greater in the former than in the latter, yet the latter contains a not inconsiderable amount. Thus in the experiment given below an acid extract of the gastric mucous membrane is made in the proportion of 1 grm. of dried tissue to 2,000 cub. centims. of hydrochloric acid—0·2 per cent. Of this acid-extract 18 cub. centims. are added to 2 cub. centims. of swollen

carmine-fibrin. Without warming a coloration = VI. of GRÜTZNER's scale is produced in eight minutes, and 1½ cub. centims. of the fibrin is dissolved in three-quarters of an hour.

Moreover, if the pepsin were simply absorbed, how could we explain the enormous difference in pepsin-content which exists between the latter part of the mucous membrane with oxyntic glands, and the adjoining mucous membrane with pyloric glands?

For whilst the acid-extract of the mucous membrane of the median portion of the stomach produces a coloration = VI. of the scale in eight minutes, a similar extract of the pyloric mucous membrane does not produce a trace of coloration with 2 cub. centims. of carmine-fibrin in three-quarters of an hour.

In favour of the view that the gastric glands do not produce pepsin, SWIECICKI adduces the following experiment:—He ligatured the œsophagus, and introduced into the stomach bits of flesh through an opening in the duodenum; in twenty-four hours the flesh was not digested, and contained only traces of pepsin.

The experiment seems to me to contain many sources of error; the ligaturing of the œsophagus seriously interferes with the peristaltic movements of the stomach; nothing is said of collecting the jelly-like masses of mucin which are secreted after such an operation, and which contain the greater part of the pepsin.

I have not repeated this experiment, because of the difficulty of removing completely the contents of the stomach and the consequent impossibility of deducing anything from a positive result were it obtained. But the following experiments, I think, show clearly that the gastric glands do form pepsin. Having destroyed the forebrain in a Frog, I lay bare the stomach, and ligature it a little below the cardia, then cut open the stomach longitudinally, and remove all mucus and fluid from the mucous membrane. On the mucous membrane a piece of sponge is then placed. Several Frogs are treated in the same manner. In two, four, six, and eight hours respectively the sponge and the mucus that has been secreted by the stomach is tested for pepsin. Within certain limits the longer the sponge has been left in contact with the mucous membrane the greater is the amount of pepsin found.

The oxyntic glands, then, form pepsin. We have seen that the oxyntic glands contain granules, although smaller than those contained by the œsophageal glands. Have the granules in the former a connexion similar to that which exists in the latter? There seems to me to be little doubt that they have. Reasons exactly similar to those which lead us to refer the ferment produced by the pancreatic and œsophageal gland-cells to the granules contained by them lead us also to refer to the granules of the oxyntic gland-cells the ferment produced by the oxyntic glands. As in the œsophagus so in the stomach—the fewer and smaller the granules contained by the cells the less is the amount of ferment contained by a definite weight of dried mucous membrane.

There is another point that deserves mention. We have seen that the anterior oxyntic glands contain, as a rule, somewhat larger granules than the posterior, and,

further, that in some of the former, granules like those of the œsophageal glands not infrequently occur. If, then, the granules are connected with the formation of ferment, we should expect a definite weight of the anterior gastric region to contain more ferment than an equal weight of the median gastric region. This, in fact, is the case (see experiment given below); part of the difference in amount is, however, probably due to the difference in the size of the glands in the two regions.

Lastly, we can consider the pyloric glands. The acid extract of the dried mucous membrane does contain some pepsin, for it dissolves carmine-fibrin more quickly than hydrochloric acid alone; but the amount is very small, and we know that a small amount of pepsin is found in nearly every tissue. I have not compared the ferment-content of the pyloric region of the stomach with that of other tissues, since it seemed to me that ... if any pepsin is produced by the pyloric glands ... amount must be so small as to be unimportant in digestion. There seemed to me to be no reason for ascribing any special function of producing pepsin to the pyloric glands in the Frog, and consequently no reason for ascribing any such function to the cells of the necks of the oxyntic glands.

The following experiment will show the method I have used in determining the pepsin-content of the œsophagus and different portions of the stomach:—

Experiment.—Frog killed by destroying brain and spinal cord. Œsophagus and stomach removed, cut open longitudinally, and pinned down with the mucous membrane uppermost. A moist sponge is passed once over it, starting from the pyloric end, and thus the greater part of the mucus or other stomach contents removed. To remove the remaining fluid or mucus the surface of the mucous membrane is carefully pressed with blotting paper, then moistened with salt solution, and again pressed with blotting paper.

The œsophagus and stomach are then spread out on a glass slide with the mucous membrane downwards and the muscular coat removed. The character of the glands is examined under the microscope and the intermediate regions between the different kinds of glands are cut away—*i.e.*, the junction of the œsophageal and gastric glands is removed, and that region which contains both oxyntic glands and pyloric glands. The oxyntic gland region is then cut through transversely into two, as nearly as possible, equal portions. Thus we obtain the mucous membrane of

(1) Œsophagus.
(2) First portion of stomach.
(3) Second portion of stomach, containing no pyloric glands.
(4) Third portion of stomach, the pyloric region containing no oxyntic glands.

The glass slide with these four pieces of mucous membrane is put in a warm chamber at about 33° C. for one day, and then kept till required in a bell jar over strong sulphuric acid.

The œsophagus and stomach of several Frogs are treated in the same way. Of (1), (2), and (3), 0·03 grm. is weighed out; of (4) 0·015 grm.*

To each of (1), (2), and (3) 3 cub. centims. of hydrochloric acid (0·2 per cent.) is added, to (4) 1·5 cub. centim. They are then put in small stoppered bottles, and left for twenty-four hours at 33° C.

To (1), (2), and (3) 9 cub. centims. hydrochloric acid (0·2 per cent.) is added, to (4) 4·5 cub. centims. Each is well shaken up and filtered.

* Of the Frogs I have used, two give about 0·03 grm. of (1), three give about the same weight of (2) and (3), nine to twelve give about the same weight of (4).

Four equal-sized test tubes are taken; in each is placed 15 cub. centims. of hydrochloric acid (0·2 per cent.), together with 2 cub. centims. of carmine-stained fibrin swollen to a jelly by hydrochloric acid (0·2 per cent.). To these 3 cub. centims. of the filtered fluid from (1), (2), (3), and (4) respectively are added; in five minutes the test tubes are shaken well, and the fibrin allowed to settle. In eight minutes the coloration is compared with GRÜTZNER's scale freshly made.

(1) tint considerably deeper than x. of scale.
(2) =viii.
(3) =vi.
(4) =0.

There is an obvious difference in the amount of fibrin undissolved in (2) and (3).

In three-quarters of an hour the following is the state of things:—

(1) Mere trace of fibrin undissolved.
(2) Small quantity fibrin undissolved.
(3) About twice as much fibrin left as in (2).
(4) No trace of coloration.

This shows clearly enough the difference in pepsin-content of (3) and (4). In order to bring out more clearly the difference in (1), (2), and (3), a smaller quantity of the acid extract must be taken.

3.20.—One cub. centim. of (1), (2), (3), and (4) is added respectively to four test tubes, each of which contains 15 cub. centims. hydrochloric acid (0·2) and 2 cub. centims. swollen carmine-stained fibrin.

3.27.—(1)=v.	3.37.—(1)=vii–viii.
(2)=i.	(2)=iii.
(3)=0.	(3)=0.
(4)=0.	(4)=0.

3.45.—(1) Nearly all fibrin dissolved.
(2) About half fibrin dissolved.
(3) Small amount dissolved.
(4) No fibrin dissolved.

At 4.45 (4) still shows no coloration. On the next day, however, only a trace of fibrin is left. Under similar circumstances hydrochloric acid (0·2 per cent.) alone takes two days to produce a like effect.

SWIECICKI gives the following table (s. 450):—

Rana temporaria; hungry; winter time.

Time	2.35.	2.45.	2.50.	2.55.	3.0.
Œsophagus	>I.	I.–II.	III.	IV.–V.	V.
Cardiac region	0.	I.	I.	II.	<II.
Pylorus	0.	0.	>I.	I.	II.

So that in twenty-five minutes the pyloric region has dissolved very nearly as much fibrin as the cardiac region, and the cardiac region about half as much as the œsophagus; and yet SWIECICKI concludes that the stomach forms no pepsin. It will be seen that my experiments show much wider differences in the pepsin-content of the various portions, in great part, I think, since I have been careful to obtain each gland form without any admixture of the neighbouring gland forms. It is possible, too, that the winter-frog used by SWIECICKI contained very few granules in the œsophageal glands—*i.e.*, that the pepsin-content was in these glands below normal.

BUFO VARIABILIS.

Histology of the Œsophageal and Gastric Glands.

Much less attention has been given to the structure and arrangement of the pepsin-forming glands in the Toad than to their structure and arrangement in the Frog. Swiecicki* apparently considers that in both these points the two are alike. He treats the question, however, very briefly, simply mentioning at the end of his paper on the œsophageal and gastric glands in the Frog, that as regards the estimation of pepsin, he obtained in the Toad results similar to those obtained in the Frog.

Partsch† denies that any glands resembling the œsophageal glands of the Frog are to be found in the œsophagus of the Toad. He found in the latter no pepsin-forming, but only mucous glands.

My own observations lead me to take up an intermediate position between these observers. Of the œsophageal glands of the Toad a very considerable number are, it is true, mucous glands, but pepsin-forming glands also occur; these, however, differ in many points from the pepsin-forming glands of the œsophagus of the Frog.

In the Toad it is less easy than in the Frog to tell exactly where the œsophagus ends and the stomach begins, for there is no constant constriction between the two regions, nor is there any change in the character of the glands sufficiently abrupt to enable a distinction of œsophageal and gastric regions to be drawn, and, as was observed by Partsch, the cylindrical cells of the surface of the œsophageal mucous membrane are devoid of cilia.

There is, however, one means of distinguishing the œsophagus from the stomach, viz., by observing the closeness of attachment of the muscular and mucous coats. In the stomach the two coats can be much more easily separated than in the œsophagus. It is by the use of this method that I have judged where the œsophagus ends and the stomach begins. It seems to me that if this method is rejected there is no alternative but to consider all the glands, mucous glands included, as occurring in the stomach, for there is no abrupt change in structure in passing from mucous to pepsin-forming glands.

The question is, however, not an important one, the important question being, Is there any differentiation in the structure and function of the glands, such, for instance, as occurs in the Frog?

In examining sections of the mucous membrane which has been treated with osmic acid, proceeding from the beginning of the œsophagus onwards, we first find short simple mucous glands as described by Partsch, then in the mucous glands are found one or two cells containing a few large granules; further on these cells become more frequent and contain more granules, until we come to the regular oxyntic glands; in these the body is usually long, and frequently coils or branches at its end; the mucous cells are confined to the necks of the glands. The granules contained by any one cell

* *Op. cit.* † *Op. cit.*

vary not inconsiderably in size; but apart from this there is a considerable diminution in the size of the granules in passing from the beginning to the end of the oxyntic gland region. The latter portion of the œsophagus contains the majority of the glands with large granules. The difference in the size of the granules in the anterior and posterior pepsin-forming regions is much less in some Toads than in others; there is also some variation in the extent to which the glands with granular cells stretch into the œsophagus. The cell-substance of the glands in the resting state stains very slightly with osmic acid. The pyloric glands resemble the pyloric glands of the Frog; the fully-formed mucous cells are, however, usually more numerous.

It is not so easy in the Toad as it is in the Frog to observe the glands in the pinned-out mucous membrane owing to the greater amount of sub-mucous tissue. In a small Toad, however, it can be seen that the latter œsophageal and anterior oxyntic glands contain in the fresh state obvious granules; they are less conspicuous than the granules in the œsophageal glands of the Frog, but much more conspicuous than those in the oxyntic glands of the Frog. Towards the pyloric region the glands become less granular and more transparent, and resemble fairly closely the Frog's oxyntic glands; the lumen, however, is frequently distinct. The pyloric glands need no special notice.

Changes which occur in the Œsophageal and Gastric Glands.

During digestion the changes which are common to the œsophageal and oxyntic glands of the Frog occur also in all the granular pepsin-forming glands of the Toad; the granules become smaller and diminish in number and size, the cells become smaller, and the cell-substance stains more deeply with osmic acid. Apart from this, there is in the Toad a slight gradual alteration in the character of the changes in passing from one end of the pepsin-forming region to the other.

In the pepsin-forming œsophageal glands we find a very feeble picture of what happens in the œsophageal glands of the Frog. During activity the outer portions of the cells become more sparsely granular, though very seldom showing an outer clear zone.

In the anterior oxyntic glands of the stomach the gland-cells are as a rule equally affected throughout; there is no alteration in the relative distribution of the granules.

In the posterior oxyntic glands the granules disappear rather more from the inner than from the outer part of the cells. An inner non-granular border is however seldom so distinctly seen as it is in the corresponding glands of the Frog.

In the Toad then the changes which the cells of the different parts of the pepsin-forming region undergo in digestion are much less divergent than they are in the Frog; moreover, in the Toad there is a gradual alteration in the character of the changes from the beginning to the end of the pepsin-forming region, whilst in the Frog the intermediate forms are largely confined to the junction of the œsophagus and stomach.

During the latter part of the digestive period there is in the Toad as in the Frog, a more or less complete restoration of the normal quiescent state of the gland-cells. The restoration, however, does not take the same course in all animals; usually as the cell increases in size there is a corresponding increase in the number of the granules; this is not always the case, for in some Toads, especially in the gastric oxyntic glands, the formation of granules does not keep pace with the growth of protoplasm.

One other variation on the normal course of events is worth mention: occasionally there is no perceptible change in the anterior gastric glands during the first three or four hours of digestion; since it can scarcely be doubted that the glands secrete during this time, we are led to infer (cp. below, p. 704) that the apparent absence of change [illegible] with the excretory processes.

Fasting causes changes like those which occur in the first stage of digestion, with the exception that the mucous cells of the necks of the glands become more instead of less prominent. This is perhaps only the case when the fasting is not too prolonged. The posterior oxyntic glands are usually more affected than the anterior.

Feeding with sponge does not cause very great changes in the œsophageal or gastric glands, although a very acid secretion with a high peptic power is obtained: the cell-granules become smaller, but I have not observed that they are more affected in one part of the cell than in another.

The Pepsin-Content of the Œsophageal and Gastric Glands.

As in the Frog so in the Toad, the amount of pepsin contained by a definite weight of dried mucous membrane is less as the cells become less and less granular during digestion. I have made no experiments to determine the pepsin-content of the mucous membrane in successive hours after feeding, but have analysed for pepsin only such cases in which I could be certain of a difference in the amount of granules contained by the respective gland-cells.

I have also compared the amounts of pepsin contained by the different parts of the œsophagus and stomach. The method of proceeding was like that described above in the case of the Frog. The amount of pepsin contained by the parts was found to vary directly with the amount of granules contained by them. The latter part of the œsophagus and first part of the stomach contains most pepsin; the amount of pepsin is rather less in the succeeding part of the stomach; the posterior gastric region in which these glands are not markedly granular in life contains considerably less pepsin than the preceding; the pyloric region contains only a very trifling quantity.

TRITON TÆNIATUS.

Histology of the Gastric Glands.

The pepsin-forming glands of the Newt may be conveniently divided into anterior oxyntic glands, posterior oxyntic glands, and pyloric glands. There is no abrupt transition in passing from the anterior to the posterior oxyntic glands, although the extreme forms of the two differ not inconsiderably.

The oxyntic glands are of the compound tubular type, several secreting tubes coming off from a common neck. This form is most pronounced in the anteriorly-placed glands. Klein* described a ring of acinous glands as occurring just above the cardia; this was confirmed by [illegible]. These "acinous" glands are the most anterior of the oxyntic glands in which the compound tubular form is most developed; they occur, as was mentioned by Sewall and myself,‡ under a ciliated epithelium.

The glands of the anterior oxyntic region occur in groups separated from one another by a considerable amount of connective tissue; this is most marked at the cardia, and becomes less and less towards the posterior oxyntic region, where the glands are not obviously arranged in groups, and are separated by only a small amount of connective tissue. In passing from the anterior to the posterior region, the mucous cells diminish in number, and the diameter of the glands decreases; the glands are longest in the first part of the posterior oxyntic region.

All the oxyntic glands both in the fresh condition and when treated with osmic acid are granular throughout. In osmic acid specimens the nuclei of the cells here and there appear as a clearer spot through the granules, but they do not form a marked feature in the glands; the cell-substance is scarcely at all stained (Plate 78, fig. 2). The posterior oxyntic glands stain somewhat darker with osmic acid than the anterior; the cell-granules contained by the former are smaller than those contained by the latter (Plate 3, fig. 3). The granules behave with reagents much as do the granules of the œsophageal and gastric glands of the Frog. Hydrochloric acid 0·4 per cent. causes them to disappear suddenly without any progressing diminution in size. When treated with bile they become smaller and smaller until they disappear. Alcohol, even 50 per cent., partly dissolves them, leaving a small slightly refractive mass behind. Some of the granules are much more readily acted on by the above reagents than others. Salt solution up to 20 per cent. leaves them unaffected.

The pyloric glands seldom contain mucous cells. They are transparent in the living condition; treated with osmic acid they remain homogeneous and stain yellow-brown. The nuclei are large compared with those of similar cells, and show when treated with appropriate reagents—as do all the gastric gland cells of the Newt (Klein)—a very

* Klein, Stricker's Hdb., vol. i., p. 542.
† *Op. cit.*, s. 198.
‡ *Op. cit.* p. 290.

distinct network. During digestion the cells of the pyloric glands frequently bulge in a very marked manner at their outer border; but from the inconstancy with which the bulging occurs I am not inclined to consider it a genuine result of secretion.

Changes which occur in the Oxyntic Glands.

The *effect of fasting* is more marked in *Triton tæniatus* than in any other animal which I have examined. In the anterior oxyntic glands a more or less distinct outer non-granular zone is formed: in the posterior oxyntic glands the clear zone is less distinct, but other changes are obvious; the cells are smaller, the lumen usually distinct, and the cell-substance stains more deeply with osmic acid. The mucous cells in the necks of the glands increase in mucigen—at any rate up to the time when the outer zone begins to be distinct in the anterior oxyntic glands.

Changes during digestion.—The diminution of granules which takes place in the living glands of a Newt after feeding it, has been described by Sewall and myself.* This diminution can also be seen in osmic acid specimens of the gastric mucous membrane, in such specimens some other changes can be seen which are less obvious in the fresh tissue. The anterior differ somewhat from the posterior oxyntic glands in the changes they undergo; in the former the outer clear zone is usually more distinct during digestion than in the latter, and in the latter the increase of the lumina and decrease in the size of the cells and granules is usually more distinct than in the former. These changes go on in all the glands, but to an unequal degree in the two glandular regions.

When the living glands are examined in the first or second hour of digestion the granules frequently appear at first sight larger than normal;† this is, I think, only caused by many of the granules having disappeared, so that instead of a confused granular mass, the separate granules can be distinctly seen. Osmic acid specimens show that at any rate about the second hour the granules are smaller. It is noteworthy that as the granules become smaller during cell activity they vary much more in size than do the granules of the resting gland (see Plate 78, figs. 3 and 4, 2 and 6). The extent and time of the changes vary widely under varying circumstances. In a normal hungry Newt fed with a small worm the following is the ordinary course of events as followed in successive osmic acid preparations.

A thinning of granules at the outer border of the cells is visible in half-an-hour to one hour; the nucleus and cell-substance begin to stain slightly. Usually at the end of an hour a decrease in the size of the cells and the cell-granules is to be seen.

These changes then proceed rapidly, the time of maximum change being three to four and a-half hours from the beginning of digestion. The cells and cell-granules are then very distinctly smaller (see Plate 78, figs. 3 and 4), and a more or less distinct

* *Op. cit.*

† They were so described by Sewall and myself from the examination of fresh specimens.

outer clear zone is present. Under ordinary circumstances the granules of the inner zone are not massed around the lumen but spread out through the inner two-thirds or even more of the cells; different glands vary widely in this respect. The small size of the granules at the time of maximum change often makes in the living glands the outer zone appear larger than it really is.

In about four and a-half hours the cells begin to return to their normal state. The changes above described disappear with astonishing rapidity, and in six to eight hours the granules are large and stretch to the outer border of the cells. The anterior glands, indeed, then differ very slightly from normal "hungry" glands. What difference still exists gradually disappears during the remainder of the digestive period. The recovery of the normal condition proceeds more slowly in the posterior than in the anterior oxyntic glands.

Since it takes twelve to twenty-four hours for the stomach to empty itself, it is obvious that the regenerative processes go on very actively at a period when the secretory processes are also still active.

The time and the extent of the several changes are very greatly affected by the condition of the animal and the amount of food given to it.

Thus, if it is fed with several worms instead of with one, not only are the changes more pronounced, but they also continue much longer; so that in twenty-four hours from the beginning of digestion the gland-cells may be somewhat small and darkly stained and possess comparatively few and small granules.

A like alteration in the extent and duration of the changes is produced by feeding a Newt which has long fasted. At the time of maximum change after such treatment the alteration from the normal "hungry" state is most remarkable. The great majority of the glands are devoid of granules and devoid of mucous cells as such; the gland-cells are small, the greater part of the cell being taken up by the nucleus; the cell-substance stains deeply with osmic acid. Plate 78, fig. 5, represents an anterior oxyntic gland in this condition. The amount of change which takes place here suggests that in the fasting animal the formative processes go on more slowly than normally (see Frog, p. 677).

Another interesting variation commonly occurs when the food for some reason or other is digested with unwonted slowness. The rapid diminution in the size of the cells during the first four hours of digestion does not take place, but the cells diminish in size and the granules in size and number for many hours, the former more slowly than the latter; it is under such circumstances that the distinction of the outer and inner zone is most plainly seen.

The question of an increase of granules during the first hours of digestion is in much the same state as the similar question with regard to the œsophageal glands of the Frog (see p. 673). Apparently in some Newts, the oxyntic glands of which are already less granular than normal, a preliminary increase takes place, but in the normal hungry Newt I have failed to observe any such increase. It will be noticed,

too, that the time during which it can take place is shorter even than in the Frog, for in the Newt a distinct decrease is generally obvious in half an hour to an hour. Moreover, in some cases of fasting Newts, *i.e.*, Newts with gastric glands diminished in size and containing granules diminished in number, I have observed a distinct diminution in a quarter of an hour which steadily progressed up to six hours.

The *effect of sponge feeding* on the oxyntic glands is much more marked in the Newt than in the Frog. There are wide variations in the extent and duration of the changes depending on the condition of the animal. But apart from this, the character of the change produced tends towards one of two types: the first, a diminution in the size of the cells and in the number and size of the granules without a very distinct formation of inner and outer zones; in the second, a distinct formation of zones without any very great diminution in the size of cells and with only a moderate diminution in the size of the granules. It will be observed that these types correspond respectively to (1) the type of change in normal rapid digestion; (2) the type of change with abnormal slow digestion. That which chiefly affects the character of the changes in the glands is, so far as I have observed, the amount of sponge given,* *i.e.*, the strength of the stimulus. In both cases the changes proceed very slowly, the first effect being obvious in about four hours, and steadily progress for one to two days; about this time the glands begin slowly to recover. Sometimes after feeding a Newt with a relatively large quantity of sponge the glands lose all or nearly all their granules and approximate in appearance to fig. 5, Plate 78. This is especially the case if the Newt has fasted for some time previously. When the sponge is removed the glands recover, although not very rapidly, their normal appearance. Glands in which the first type of change has occurred form small granules which soon stretch throughout the cells; the cells and their granules then together become larger. Plate 78, fig. 7, shows the first stage of this return; the granules are small, and though absent from the outer part of the cell do not form a dense zone around the lumen. (Dark clumps of fat globules are seen at the periphery, cp. p. 670.) Glands in which the second type of change has occurred continue to show distinct zones for some hours after the granules have begun to increase. Plate 78, fig. 6, shows the condition of the anterior oxyntic glands in a Newt which had been fed for eighteen hours with sponge, and the stomach of which was put in osmic acid six hours after the removal of the sponge.

The Pepsin-content of the Gastric Glands.

In the same manner as that given above for the stomach of the Frog I have analysed the amount of pepsin contained by the anterior oxyntic, the posterior oxyntic, and the pyloric glands in *Triton tæniatus*.

The amount of pepsin contained by the pyloric glands is very small; in both oxyntic

* I may mention that the most marked instances of the second type of change I obtained in February, 1880; of the first type of change in the autumn of 1880.

regions a considerable amount is found. It will be remembered that in the oxyntic glands alone obvious granules occur in the fresh state; hence, then, pepsin is found in quantity only where obvious granules occur.

A definite weight of gastric mucous membrane taken from the posterior oxyntic region contains a somewhat greater quantity of pepsin than an equal weight of mucous membrane taken from the anterior oxyntic region. We have seen that in the latter the glands are shallower and are separated from one another by much more connective tissue than in the former, or that a definite weight of mucous membrane taken from the anterior region contains a less amount of granules than an equal weight taken from the posterior region. Hence probably the difference in the pepsin-content of the two parts.

In the Newt, as in the Frog and Toad a diminution in the cell-granules, such as occurs in fasting or during digestion, is accompanied by a diminution in the amount of pepsin contained by a definite weight of mucous membrane.

TRITON CRISTATUS.

The gastric glands of *Triton cristatus* are in the resting state very like the resting gastric glands of *Triton tæniatus*. There is, however, much less difference between the glands found in the anterior and posterior oxyntic regions in the former than in the latter. The anterior oxyntic glands of *Triton cristatus* are in groups and consist of several tubes coming off from a single duct, but the granules contained by the cells are as a rule not much larger than those contained by the cells of the posterior oxyntic glands. Further, the digestive changes in the glands are much less marked in *Triton cristatus* than in *Triton tæniatus*. Even at the period of maximum change with worm- or sponge-feeding it is rarely that an outer non-granular zone occurs. In one or two glands, usually either at the beginning or at the end of the oxyntic region, the outer portion of the cells does not contain granules, but even then the line between the granular and the non-granular zones is not a sharp one. Both in the fresh and in osmic acid specimens fewer granules are found towards the outer portion of the cells during digestion; there is a diminution in the size of the cells and some increase in the diameter of the lumina. In the latter period of digestion these changes are more or less completely repaired.

We have seen that during fasting a clear zone gradually appears in the oxyntic glands of *Triton tæniatus*, in the glands of *Triton cristatus* I have not seen a corresponding change under similar circumstances.

The time of maximum change with sponge-feeding is in the animals I have observed about twenty hours. Osmic acid specimens then show that the cells are much smaller than normal; the cell-substance stains much darker; the granules in the cells are also fewer and smaller; the granules are most numerous in the inner portions of the cells but are seldom absent from the outer portion. The lumen is enlarged often con-

siderably. Of the pyloric glands little need be said; like the pyloric glands of *Triton tæniatus*, they do not contain any definite mucous cells.

Pepsin-content of the gastric glands.—What has been said (p. 689) above on the pepsin-content of the gastric mucous membrane of *Triton tæniatus* holds also for that of *Triton cristatus*, with one or two slight modifications.

The difference in the pepsin-content of the anterior and posterior oxyntic regions is greater in the large than in the small Newt. This is, as before, in correspondence with the relative amount of granules contained by the two parts. It will be remembered that in *Triton tæniatus* the granules are considerably larger in the anterior than in the posterior oxyntic glands, and that in *Triton cristatus* the difference in the size of the granules in the two regions is only slight. The less frequency of the glands in the anterior region is, then, more compensated in the small than in the large Newt.

We have seen that there is less disappearance of granules during digestion in *Triton tæniatus* than in *Triton cristatus*; we find also that in the latter there is a less difference in pepsin-content in hunger in digestion than in the former.

COLUBER NATRIX.

Histology of the Gastric Glands in Hunger and Digestion.

The end of the œsophagus and the beginning of the stomach can be readily distinguished; in passing from one to the other the mucous membrane becomes suddenly thicker and more opaque; the junction line of the two is not a circle, but an oval, the glands occurring first on the ventral side of the alimentary canal.

The epithelium of the œsophagus near the stomach consists of long and unusually thin cylindrical cells, which have at their free ends suffered some amount of mucous metamorphosis. Partsch* described the epithelium near the stomach as consisting of ciliated and goblet cells; I have not observed either of these. There are no proper œsophageal glands; there are some few dippings down of the surface epithelium, but the cells do not markedly change in character.

The most anterior glands are, as in the Newt, separated by more connective tissue than are those in the remainder of the stomach. They are arranged in groups, and several tubes are connected with each neck; the tubes, moreover, usually divide in their course. In passing towards the pylorus the glands consist of a smaller number of tubes, but remain, as a rule, complex glands.

In the fresh state all the oxyntic glands are densely granular, and little difference is to be seen in them except that the most anterior are larger in cross section. The pyloric glands are transparent and non-granular; the intermediate region is comparatively large.

If three Snakes are fed each with a Frog, and the gastric glands are examined in

* *Op. cit.*

the fresh state in one, two, and three days respectively, it is seen that the changes which take place in the gland-cells are first obvious in the posterior oxyntic region, and steadily progress towards the anterior region. No zones are at any time distinctly formed, but the granules contained by the cells become fewer and less obvious; so that the posteriorly-placed oxyntic glands appear in the fresh state nearly as devoid of granules as the pyloric glands. The two gland-forms can be, however, still distinguished; they have a different general look. Compared with the pyloric glands, the oxyntic glands have a faint yellow tinge and almost oily appearance; sometimes, too, the rounded outline of their cells can be made out.

The greatest change that I have observed was in a Snake examined sixty hours after feeding. The glands in the latter third of the oxyntic region showed scarcely any granules; in the middle third the glands contained many less granules than normally; they contained, however, more and more in passing towards the anterior oxyntic region, where the glands were densely crowded with granules. That is, the changes increase in intensity in passing from the beginning to the end of the oxyntic region. This is very similar to the manner in which the glands of the Rabbit's stomach are affected during digestion.

In osmic acid specimens some other points can be made out: The glands become longer and narrower in passing from the beginning to the median portion of the stomach; thence to the pyloric region they become shorter—at first gradually, then more rapidly. In the Snakes which I have examined the pyloric glands are simply mucous glands; the sub-cubical cells which form the sole constituent of the pyloric glands in Newts and a partial constituent of the pyloric glands in Frogs are absent in the Snake. Partsch,* however, found exactly the contrary: sub-cubical cells he observed, but no mucous cells.

Osmic acid specimens of the gastric mucous membrane of a hungry animal show no great difference in the characters of the cells throughout the oxyntic gland region; all contain granules fairly equally distributed throughout their substance. The granules vary in size; the larger ones closely resemble those in the œsophageal glands of the Frog or in the anterior oxyntic glands of the Newt. The granules are, as a whole, largest in the most anterior oxyntic glands, and, generally speaking, the farther the glands are from the beginning of the stomach the smaller are the granules they contain. The cell-substance stains very slightly, but more in the posterior than in the anterior oxyntic glands. The necks of the glands, especially of those in the median gastric region, differ from the necks of the gastric glands in other animals; sub-cubical cells are usually absent.

When osmic acid specimens of a digesting stomach (sixty hours) are examined there is little or no change to be seen in the anterior portion; the granules are large and the cell-substance unstained (Plate 78, fig. 8). In passing towards the latter third of the gastric gland region certain changes become more and more obvious: the granules

* *Op. cit.*, s. 200.

become smaller and fewer, the cell-substance begins to stain, the cells are smaller, and the lumen often visible (see Plate 78, figs. 9, 10). These changes are most pronounced in the latter third of the oxyntic gland region. The lumen is never large; it may be either straight or zig-zag, depending on the position of the cells in the gland tube. It is to be remarked that osmic acid brings out a number of granules in cells which in the fresh condition showed scarcely any. In this respect osmic acid specimens show a less striking difference between the anterior and posterior oxyntic glands than is shown by the fresh specimens.

I have not noticed any great change from normal in the glands of the fasting Snake. The cells become somewhat smaller; this is marked in the posterior gastric gland region by an increase of the lumen; the cell-substance, too, especially in the posterior gastric region, stains more deeply with osmic acid, and contains smaller granules.

In the posterior oxyntic region during digestion the granules disappear somewhat more at their inner than at their outer portion (see Plate 78, fig. 9); this is, however, so far as I have observed, much less markedly the case than in the Frog.

The Pepsin-content of the successive Regions of the Stomach of the Snake.

Having observed that in the digesting stomach the diminution in the number of the granules became more and more marked in passing from the anterior to the posterior region, I made some experiments to determine whether there was a corresponding diminution in the amount of pepsin. I divided the mucous membrane containing gastric glands into four parts, throwing away a strip between each part. The four pieces were then put aside to dry in a bell-jar containing sulphuric acid. A definite weight of the dried mucous membrane of each part was taken and treated with dilute hydrochloric acid for twenty-four hours. The details of the method of proceeding were in the main like those given above (p. 681) in the account of the pepsin-content of the stomach of the Frog.

The first experiment was on a Snake that had been fed with a Frog twenty-four hours previously. Only that part of the Frog which was in the posterior portion of the stomach had been much digested. The glands of the posterior portion of the stomach were distinctly granular, although they were, as a rule, rather more thinly granular at their outer portion. The analysis showed a steady diminution in the amount of ferment from the beginning to the end of the stomach, the first part containing more than the second, the second than the third, the third than the fourth. The fourth part contained very much more than an equal weight of the mucous membrane of the pyloric region.

The second experiment was made on a Snake sixty hours after feeding with a Frog. The legs of the Frog, which were in the anterior part of the stomach, were still nearly intact. Examined in the fresh state, the first part of the stomach showed many granules; the second part showed less than the first, but still a considerable number; the third part few granules; the last part very few.

The analysis showed that the diminution in the amount of ferment in passing from the beginning to the end of the stomach was much more rapid than in the previous case. In parts three and four a comparatively feeble pepsin action was found—in three, rather more than in four; the difference in rapidity of action between the second part and the fourth was considerably greater than between the fourth and the pyloric region. In the normal hungry Snake the amount of pepsin contained by a definite weight of dried mucous membrane also diminishes from the anterior to the posterior portion, but to a much less extent than in the digesting animal. The amount of pepsin contained by the pyloric region is very small, although, as in the Frog, Newt, and Toad, quite appreciable. In the Snake I compared the amount of pepsin contained by equal weights of œsophagus and pyloric gland region; from the former I obtained no pepsin reaction.

It will be noticed that in all cases the amount of pepsin contained by a definite weight of mucous membrane varied directly with the granularity of the gland-cells.

In the above experiments the dried mucous membrane was extracted with dilute hydrochloric acid; quite similar results are, however, obtained if the dried mucous membrane be extracted with glycerine, and the glycerine extract tested for pepsin.

I have made some experiments to determine whether the gastric glands of the Snake contain pepsinogen. Since I propose considering in detail in a later paper the question of the formation of pepsinogen in gastric glands, I will here only mention that the stomach of the Snake contains a very considerable amount of a substance insoluble in glycerine, but which when treated with dilute hydrochloric acid gives rise to pepsin, and that this substance is in greatest quantity in the anterior region of the stomach, and diminishes in amount in passing towards the posterior region.

General Conclusions on the Secretory Processes in Pepsin-forming Glands.

From the preceding details some general conclusions can, I think, be drawn.

Pepsin is formed from the granules seen in the gland-cells in the living state.

An account has been given above of the pepsin-content of the stomach and of various parts of it in *Rana temporaria*, *Bufo vulgaris*, *Triton tæniatus*, *Triton cristatus*, and *Coluber natrix*. By the method used, viz.: extraction of the dried tissue with hydrochloric acid 0·2 per cent., the total amount of free and combined pepsin* present in the tissue was obtained.

* Grützner (Neue Unters. ü. d. Bildung u. Ausscheidung d. Pepsins § 26, 1875) has pointed out that this method only gives accurate results under certain conditions, viz.: (1) when a relatively large quantity of hydrochloric acid is used for extraction; (2) when the extraction proceeds for at least ten hours; (3) when the amount of tissue is not too great compared with the amount of pepsin. The first two conditions were complied with in all my experiments. With regard to the last, it may result that the amount of pepsin contained by the pyloric glands is greater than that which I have found.

In each of the animals studied we have seen that the amount of pepsin contained by any part of the stomach (or of the pepsin-forming region) is directly proportional to the amount of granules contained by the gland-cells of that part.

In each of the animals studied we have seen that the cell-granules diminish in number and size during digestion. GRÜTZNER found in the animals* investigated by him a diminution, during digestion, in the amount of pepsin contained by a definite weight of the stomach. Without examining in detail the amount of pepsin contained by the gland-cells at different digestive stages, we have seen above in each of the animals considered that a definite weight of gastric mucous membrane taken at a stage of digestion when the granules are markedly diminished contains a markedly diminished amount of pepsin.

Further, we have seen that during fasting the number of granules contained by the gland-cells diminishes, and we know from GRÜTZNER's† experiments that during fasting the amount of pepsin contained by a definite weight of gastric mucous membrane diminishes.

Lastly, it has been shown by SEWALL and myself‡ that in the Rabbit the amount of pepsin contained by any portion of the stomach is in direct proportion to the number of granules contained by the "chief" cells of that portion in the living state.

Hence, then, in all the cases which have been investigated the amount of pepsin is in direct proportion to the amount of granules. From this, I think, we may fairly conclude that the granules in the gland-cells give rise to pepsin. We have at least as much reason to conclude this as we have to conclude that granules in the pancreatic gland-cells give rise to trypsin.

It may, however, be said that since the cells diminish in size during secretion, the diminution in the amount of ferment might be caused by a diminution in the cell-substance and not in the granules. The following consideration will, I think, show that this cannot be the case. Although each gland-cell becomes smaller, there is in most cases no obvious diminution, and in some there seems to be an actual increase in the amount of cell-substance. The removal of the granules alone is sufficient to cause the diminution observed in the size of the cell. Further, since the glands are smaller, to obtain an equal weight of resting and of digesting mucous membrane, a larger area of the latter, *i.e.*, a greater number of glands, must be taken.

* GRÜTZNER, Neue Unters., u. s. w., § 36 *et seq.*, Dog, Pig, Rabbit, and Cat. PFLÜGER's Archiv. Bd. xvi., 1878, § 118, Dog; § 120, Œsophagus of Frog.

† GRÜTZNER, Neue Unters. u. s. w., § 52, Rabbit; § 61, Dog, Cat. PFLÜGER's Archiv., Bd. xx., 1879, § 407 and 408, Œsophagus of Frog.

‡ Journal of Physiol., vol. ii., p. 291, *et seq.*, 1879. I may mention that the arrangement of the gastric glands in the Guinea-pig is similar in main points to the arrangement described by SEWALL and myself as existing in the Rabbit.

The gland-cells do not store up pepsin as such, but store up zymogen, out of which pepsin arises when the cell secretes.

The researches of HEIDENHAIN* have shown that the cells of the pancreas do not store up trypsin but store up a substance which under certain conditions can give rise to trypsin.

Since it had been shown by EBSTEIN and GRÜTZNER† that the gastric glands of Mammals contain a certain quantity of a substance capable, when acted on by sodium chloride or by dilute hydrochloric acid, of giving rise to pepsin, HEIDENHAIN introduced the word "zymogen" to include the antecedent of the ferment in both these and in any later found cases.

In the pancreas the conspicuous granules in the cells consist in part or wholly of zymogen. The zymogen can then in this instance be distinguished from the rest of the cell-substance.

When we compare in the Frog the pancreatic with the œsophageal gland-cells, and observe the close resemblance which exists between the two, especially as regards the granules they contain, it seems in the highest degree improbable that they should differ so widely as that one should store up zymogen and the other store up ferment.

We have seen that the gastric glands of the Newt closely resemble the œsophageal glands of the Frog as regards the reaction and behaviour of the granules they contain: the other gastric glands we have investigated above, differ in some points from both of these glands, but in the important point of the behaviour of their granules show a general similarity to them. In these glands then, too, the microscopical appearances seem to me to render it far more probable that the granules stored up should consist of zymogen, than that they should consist of ferment or other ready-formed secretory product.

I know of no satisfactory instance in any gland in which the substances found in the secretion are found stored up in the cells. In the liver-cells none of the characteristic constituents of bile are found. It may be objected that in these no substance is stored up, and that the cell protoplasm in activity forms straightway and excretes the constituents of bile. From many observations carried on during the past twelve months on the liver of the Frog, Toad, Newt, and Snake, I have been able to satisfy myself that in these cases also granules are stored to be used during secretion.

In mucous glands the evidence is against the substance stored by the cells being actually the mucin found in the fluid secreted. It is probably a pre-product closely related to mucin.

In the serous glands the facts known are not sufficient to be of much use for or against.

* HEIDENHAIN, PFLÜGER'S Archiv., Bd. x., s. 557, 1875.

† EBSTEIN and GRÜTZNER, PFLÜGER'S Archiv., Bd. viii., s. 122, 1874.

When the secretion produced by a gland contains undissolved particles, these particles may be stored up by the gland, thus fat globules are apparently stored up by the cells of the Harderian and Mammary glands.

I do not overlook the fact that a glycerine extract of the fresh Mammalian gastric mucous membrane contains a good deal of pepsin, but I believe this to be due simply to the zymogen being more easily split up in these glands than in others. It will be remembered that the granules of the "chief" cells which probably consist in part or wholly of zymogen are not preserved by osmic acid; *i.e.*, these zymogen granules split up with osmic acid, whilst the other zymogen granules we know of do not. Further, as mentioned above, EBSTEIN and GRÜTZNER have shown that the gastric glands of Mammals do contain some zymogen.

The general course of reasoning given above, led me to make some direct experiments on the subject. The result completely confirmed the justice of the deductions drawn from microscopical examination. Since I propose to discuss in a later paper the whole question of the formation of pre-products by gland-cells, I will here only briefly mention the main facts which show that zymogen and not ferment is stored up by the glands we are considering.

If the œsophagus or stomach of a Frog be placed in glycerine as rapidly as possible after removal from the body, the glycerine extract has only a weak peptic power.

If the œsophagus or stomach of a Frog be kept moist for twenty-four hours before it is placed in glycerine, the glycerine extract has a very much greater peptic power.

If the œsophagus and stomach which has been extracted with, say, 5 cub. centims. of glycerine for a week be washed free of glycerine and treated with 5 cub. centims. of dilute hydrochloric acid, then an enormously greater amount of pepsin is found in the acid than is found in the glycerine extract.

The glycerine extracts increase somewhat, although only slightly, in peptic power when treated with dilute acid for twenty-four hours.

Since in testing a glycerine extract for pepsin, hydrochloric acid is added, it is possible that even the small amount of pepsin found in it may arise from the splitting up by the acid of combined pepsin.

These facts, I think, show that both the œsophageal and gastric glands form zymogen which under certain conditions, and particularly when acted on by dilute acids, give rise to pepsin. The zymogen is only slightly soluble in glycerine.

Similar observations made on the gastric glands of *Triton tæniatus* give similar results. In the Snake, lack of animals has prevented me from comparing the pepsin-content of glycerine extract of the fresh with glycerine extract of the exposed stomach, but the experiment mentioned above (p. 694) shows that the gastric glands of the Snake also contain zymogen.

Since the pepsin of the œsophageal and gastric glands which we have considered arise from granules which are visible in life, the fact that the pyloric glands show no granules in life and yet apparently form pepsin may seem to require explanation.

We know that there are many cells which consist of protoplasm* and inter-proto-

* I use the word protoplasm for all living substance. I take the network which KLEIN and others have shown exists in so many gland-cells to be protoplasm, and the interfibrillar substance to be chiefly at any rate stored up material.

plasmic substance, *i.e.*, of protoplasm and of the products of its metabolism which are kept imbedded in the cell; and that further in a great number of such cells the protoplasmic and inter-protoplasmic portions are not optically distinguishable from one another in the living cells. Whether they are distinguishable or no depends mainly upon the refractive index of the inter-protoplasmic substance. The antecedent of pepsin in the above-mentioned gastric glands differs from the protoplasm sufficiently in refractive power to be visible in life.

In the pyloric glands the inter-protoplasmic substance is almost altogether mucigen, having much the same refractive power as the protoplasm which formed it. It seems, then, not unnatural that the small amount of the antecedent of ferment contained by the pyloric gland-cells should be unable to render the inter-protoplasmic substance obvious in life.

Since the pepsin of different animals differs somewhat in its properties,* it is probable that both pepsin and its antecedent differ in different animals somewhat in chemical constitution. I can then readily imagine that the gastric gland-cells of some animals *may* form an inter-protoplasmic substance giving rise to a large quantity of pepsin, and yet having so nearly the same refractive index as the cell-protoplasm as to be very slightly or not at all visible without the action of reagents.

Since the inter-protoplasmic substance gives rise to pepsin, it is in the highest degree improbable that the cell should form pepsin in any other way than through the medium of the inter-protoplasmic substance. This means in the gastric glands we are considering, that pepsin is formed in the cells only through the medium of granules. *If, then, two gastric glands secrete an equal amount of pepsin, they will have used up an equal, or, allowing for a slight difference in the chemical composition of the granules, a nearly equal, amount of granules.* This deduction we shall have occasion to allude to later.

During secretion the three chief phenomena which can be recognised in gland-cells, viz.:—(*a.*) *a using up of granules,* (*b.*) *a fresh formation of granules,* (*c.*) *a growth of protoplasm—go on simultaneously. The different aspects of the gland-cells depend upon the relative activity of these three processes.*

This view has been already put forward by HEIDENHAIN to account for the changes which take place in the pancreas. He says:—†

"In the cells a continuous change takes place; a using up of substance in the

* See HOPPE-SEYLER, Physiol. Chemie., s. 218, 1878.

† "An den Zellen findet ein fortwährender Wandel statt; Stoffverbrauch innen, Stoffansatz aussen. Innen Umwandlung der Körnchen in Secretbestandtheile, aussen Verwendung des Ernährungsmaterials zur Bildung homogener Substanz, die sich ihrerseits wiederum in körniges Material umsetzt. Das Gesammtbild der Zelle hängt von der relativen Geschwindigkeit ab, mit welcher sich diese Processe vollziehen. Die

inner portion, an addition of substance to the outer portion. In the inner portion, change of the granules into secretory constituents; in the outer portion, employment of the nutritive material for the formation of homogeneous substance, which on its part is changed into granular material. The appearance of the cell as a whole depends upon the relative rapidity with which these processes proceed. The first digestive period is characterised by a rapid using up in the inner portion and a rapid addition to the outer portion. In the second digestive period the most active changes proceed at the junction of the inner- and outer-zone, inasmuch as the substance of the latter is converted into the substance of the former."

HEIDENHAIN* draws a just analogy between the changes which take place in the pancreas and those which take place in the gastric and in the serous and mucous glands; since we now know† that in these latter the cells form granules to be used up in activity, the analogy is even closer than that drawn by HEIDENHAIN.

The changes described by HEIDENHAIN as occurring in the gastric glands differ somewhat from those which he describes as occurring in the pancreas. The differences mainly arise from his conclusions about the gastric glands having been made on alcohol hardened instead of on fresh specimens. Thus instead of speaking of the granules of the chief cells he speaks of the non-staining substance, and instead of the homogeneous protoplasm of the living cell he speaks of the cloudy or finely granular protoplasm which is seen in the cells of alcohol specimens. He says:‡ "In the earlier secretory period the in-come is as a rule in excess of the out-go in the chief cells of the glands of the fundus, hence ensues an increase in the size of the cells. At the same time however there is an active formation of non-stainable substances (pepsinogen and pepsin) from the protoplasm, hence the cloudiness of the cells is for the time only slight. As digestion proceeds the out-go becomes gradually in excess of the in-come, hence the cells diminish in size; at the same time the change of the albuminous substance still continued to be taken up by the protoplasm goes on more slowly, hence the cells become more cloudy, richer in protoplasm, and more deeply stainable."

erste Verdauungsperiode charakterisirt sich durch schnellen Verbrauch innen und schnellen Ansatz aussen. In der zweiten Periode vollziehen sich die lebhaftesten Veränderungen an der Grenze der Innen- und Aussenzone, indem die Substanz der letzteren sich in die Substanz der ersteren umwandelt."—(PFLÜGER's Arch., Bd. x., s. 569, 1875, and HERMANN's 'Handbuch d. Physiol.,' Bd. v., s. 202, 1880.)

* HERMANN's 'Handbuch,' s. 147, 1880.

† Journal of Physiol., vol. ii., p. 261, *et seq.*, 1879.

‡ "Beim Beginne der Absonderung überwiegt in den Hauptzellen der Fundusdrüsen in der Regel die Aufnahme über die Abgabe, deshalb tritt Vergrösserung der Zellen ein. Gleichzeitig aber findet noch lebhafte Bildung nicht färbbarer Substanzen (Pepsinogen und Pepsin) aus dem Protoplasma statt, deshalb wird die Trübung der Zelle vorläufig eine nur geringe. Beim Fortgange der Verdauung wird allmählich die Abgabe vorherrschend über die Aufnahme, deshalb schwellen die Zellen ab; gleichzeitig geschieht die Umwandlung der immer noch von dem Protoplasma aufgenommenen Albuminate langsamer, deshalb werden die Zellen trüber, protoplasmareicher, und stärker färbbar."—(HERMANN's 'Handbuch d. Physiol.,' Bd. v., s. 146, 1880.)

He points out* two differences between the changes which occur in the pancreatic and gastric glands. First, that the regeneration of the used-up material takes place in the pancreas whilst the gland is still secreting, but does not in the gastric glands. Secondly, that in the pancreas the granules are formed out of the homogeneous outer zone, and in the gastric glands out of the finely granular cell protoplasm. As to the first point of difference, we have seen that in the Frog, Newt, and Toad the used-up material is regenerated more or less completely whilst secretion is still going on. The second point of difference chiefly rests on the assumption that the protoplasm of the fresh cell is finely granular, since in alcohol specimens an increase of protoplasm in the cell is marked by an increased cloudiness.

The differences which exist with regard to the formation of granules in those cells which acquire an outer non-granular zone in activity and those which do not we shall discuss later. We have first to consider what facts we have to support the view given above as to the changes which take place in the pepsin-forming glands during secretion.

We have to show that (*a*) the using up of granules, (*b*) the formation of granules, and (*c*) the growth of protoplasm all go on from the beginning to the end of secretion.

(*a*.) Since it has been shown that the granules give rise to pepsin, and since we know of no other origin for pepsin, we may conclude that as long as pepsin is formed, *i.e.*, as long as the secretion goes on, granules are used up.

(*b*.) We have seen that in all cases in the latter half to two-thirds of the digestive period the granules increase. There can be no doubt that during this time a secretion is going on; this is indeed shown by feeding a Frog or Newt with sponge. It is then seen that the simple mechanical stimulation caused by the presence of a foreign body in the stomach is sufficient to cause a secretion lasting for two or more days. HEIDENHAIN, too, has shown that in the Mammal secretion of gastric juice goes on as long as food remains in the stomach. But since a secretion means a using up of granules, there is during a large part of the digestive period a using up and formation of granules going on at the same time.

We have also seen that in certain circumstances an increase of granules occurs in the first hours of digestion. During this time there is also a using up of granules, that is, the two processes go on together, sometimes at any rate in the first hours of digestion.

Now, when we consider that the using up of granules is continually going on, and that the formation of granules certainly takes place in the latter digestive period and sometimes in the first digestive period, and further consider that the formation of granules can only become obvious in the glands when it takes place at a faster rate than the using up, the conclusion seems irresistible that there is a formation of granules during the whole digestive period, but that during the first five to ten hours it is not sufficiently rapid to cover the loss from the using up of granules. In certain circum-

* HERMANN'S 'Handbuch d. Physiol.,' Bd. v., s. 202, 1880.

stances the loss may be, though it is not normally, covered during the first one to three hours. The number of granules contained by any gland-cell at a given time during secretion depends then upon the relative rate of formation and using up of granules.

(*c.*) There is, I think, also sufficient evidence to justify us in believing that there is a continuous though not uniform growth of protoplasm during activity. The organic substances in the fluid secreted are formed, for the most part at any rate, by the cell-protoplasm. A growth of cell-protoplasm sufficient to cover the loss of the organic substances secreted must then, at some time or other, take place. Does this growth take place in "rest" or "activity"? The following consideration will, I think, show that although there may be a slight growth in rest, the growth takes place, in the main, during activity. With the exception, perhaps, of the oxyntic glands of the Snake, all the glands above considered have normally nearly returned to their hungry condition by the end of the digestive period, *i.e.*, at the end of twelve to twenty hours of secretion. The protoplasm which has been continuously diminishing, owing to the continuous formation of zymogen, is hardly appreciably in less quantity than at the beginning. The protoplasm, then, has grown during secretion. I know of nothing which tends to show that the growth of protoplasm is limited to any particular period of digestion; there are, on the other hand, several facts which go far to prove, if indeed they do not prove, that the growth is continuous during digestion.

In the second stage of digestion the cells increase in size, since we have no reason to imagine that the stored-up substances, the zymogen and so forth, occupy a greater space than the protoplasm from which they are formed, we may, I think, fairly refer the greater part at any rate of this increase in the size of the cells to a growth of the protoplasm; further, since the cells are during this period still losing substances to the fluid secreted, the total growth of protoplasm must be very considerable.

We have seen that under certain circumstances it may happen that there is a similar increase in the size of the cells, *i.e.*, a similar growth of protoplasm in the first one or two hours of digestion; we might then adopt a course of reasoning like that adopted in considering the formation of granules, and conclude that the protoplasm is growing also in the first stage of digestion when the cells are diminishing in size. We have, however, more direct evidence to show that this is the case.

It is true that the cells diminish in size during this stage of digestion, but they do not diminish sufficiently in size to warrant us in supposing that there is no growth of protoplasm. The mere extraction of the granules which have disappeared would leave the cells very much smaller; this fact becomes the more striking when we reflect that in the meantime more protoplasm has been used up to form granules. The changes which take place in the œsophageal glands of the Frog after the animal has been fed with sponge will serve as an instance. When the granules have entirely disappeared from the cell, is it to be supposed that the cell is simply as it was at the beginning of secretion, except that it has lost all its granules and a part of its protoplasm, viz.: that

which has been converted into granules. Clearly were this the case the cell would be very much smaller than it is.

Further, in the first stage of digestion the cells increase in power of staining with carmine and other similar reagents. This is only what has been shown to take place by HEIDENHAIN and others in a great number of gland-cells. The increased staining power is usually considered a proof of the increase of protoplasm. From this, too, then we should conclude that the protoplasm is growing, although the cells are diminishing in size.

There can, I think, be little doubt that these three processes also go on simultaneously in the serous glands during secretion. The increased staining power of the cell with carmine indicates that fresh protoplasm has been formed. I have previously shown that a using up of granules takes place. If we grant, as from analogy I think we may, that the granules give rise to the organic substances in the secretion, it would, as in the gastric glands, appear unlikely that these substances should be formed from the cell-protoplasm, otherwise than through the intermediate step of granules. But from this I think it follows that the granules are formed during secretion as well as in rest, for with a corresponding amount of solids excreted, the visible diminution in the number of granules is less in the sub-maxillary than in the parotid of the Rabbit, so that more new granules must have been formed by the sub-maxillary gland-cells than in the parotid. Further, the total amount of organic solids obtainable by protracted stimulation of the secretory nerves of the parotid seems to me too great to allow them to be referred to the granules present to start with.

Some remarks on the difference of rate with which the above-mentioned three processes go on in the gastric glands, according as the animal is fed with digestible or with non-digestible food.

If we compare the granular content of the gastric gland-cells at different periods after an animal has been fed with worm, with the granular content at different periods after an animal has been fed with sponge, we find that the main point of difference is that in the latter case the diminution in the number of granules lasts much less time, but is, during that time, much more rapid.

The explanation of the appearances which accompany sponge-feeding seems simple; the stimulation causes a slow nett using up of granules; after about twenty-four hours the stomach becomes accustomed to the presence of the sponge, and the stimulation becomes less and less; as the stimulation becomes less the tendency of the cells to form granules gets the upper hand of the using up of the granules, and they slowly increase.

In the worm-fed animal we have to account for the more rapid using up during the first hours of secretion, and the more rapid formation during the last hours of secretion.

HEIDENHAIN has shown that in the Mammal the presence of digestible food causes a more rapid flow of gastric juice than the presence of non-digestible food. He referred this to a chemical stimulation caused by the absorbed products of digestion. The same explanation will probably serve here. It seemed not unlikely that the peptone formed might be the particular product causing increased secretion. I have made some experiments upon the effect of injecting peptone into the lymph-sac and

into the stomach of the Frog; but the results, although on the whole favourable to the view that peptone causes a secretion, yet present some exceptions which prevent my forming any decided opinion on the subject.

The increase of granules and the growth of the cell during the last hours of digestion is in all probability due to the presence of the assimilated products of digestion.

In a worm-fed Frog, from the eighth to the eighteenth hour the secretion is going on at least as rapidly as in a sponge-fed animal, and yet the granules increase in number in the cells. We have seen reason to think that the first absorbed products cause an increased using up of granules, so that during the eighth to the eighteenth hour of digestion there must be some additional factor causing the formation of granules. It seems to me most likely that this additional factor is the presence of some of the digestive products which have been converted into a fit state to be assimilated by the gland-cells.

The effect of digestible substances in increasing the rate of growth of the cells and the formation of granules is strikingly shown in such an experiment as the following.

Three Frogs are fed with sponge. In twenty-four hours the sponge is removed from two of these Frogs; one of the two is fed with worm. All three are killed in twenty-four hours more, and the œsophageal glands examined. We have, then, specimens of glands from Frogs all forty-eight hours from the time of feeding with sponge, but (1) has been stimulated by the sponge during the whole time, (2) has been stimulated during the first twenty-four hours only, (3) during the first twenty-four hours with sponge, during the last twenty-four hours with worm.

Now, it is found that the gland-cells of (1) are small and contain very few granules, whilst those of (3) are large and contain many granules; yet the gland-cells of (3) have poured forth at least as much and probably more secretion than those of (1), *i.e.*, at least as many granules have been used up in (3) as in (1), and yet (3) contains very many more granules.

Further, the size of the cells and the number of the granules is not very different in (2) and (3), *i.e.*, the gland-cells of (2) which have used up no granules during the twenty-four hours before examination are not more granular than those of (3), in which a very considerable using up of granules has taken place.

We see from (2) that when the stimulus is removed the cells form granules, although no digestive products have been absorbed; the more rapid formation of granules which takes place when digestible substance is given I attribute to an increased supply of those substances which serve as food to the cells.

The same conclusion results from comparing the effects of sponge and of worm-feeding on *Triton tæniatus*. The details of such experiments are given above; it is unnecessary to discuss them here.

The differences in the changes produced by secretion in different gland-cells is due partly to variations in the relative rates with which the using up of granules, the growth of protoplasm, and the formation of granules go on both in each cell as a whole and in various parts of it; partly also to variations in the power of the gland-cells to move the granules towards the lumen.

We have seen that in any one form of cell the three processes go on at different rates at different periods of secretion. We should be *d priori* inclined to conclude that they would go on at different relative rates in the gland-cells of different animals. A comparison of the extent of the changes during activity presented by the gastric glands of *Triton tæniatus* and those of *Triton cristatus* makes this conclusion, I think, necessary.

If a Newt of each species be taken and each fed with an amount of worm sufficient to moderately distend the stomach, the apparent using up of granules is much greater in the gastric glands of *Triton tæniatus* than in those of *Triton cristatus*. But we have no reason to think that one has secreted proportionately more gastric juice than the other, that is to say (see above, p. 698), we have no reason to think that one has used up more granules than the other.

Since, however, we observe a marked difference in the total loss, and know further that in both the loss as it goes on is partially replaced, the conclusion naturally follows that in one the replacement is more active than in the other, *i.e.*, that the comparatively slight change observed in the gland-cells of *Triton cristatus* is due to the formative processes going on at a rate closely corresponding with that of the excretory processes. The effect of sponge feeding in the two species of Newts also leads to the same conclusion. In both cases an acid secretion of good digestive power is obtained, but the observable diminution in the number of granules in the cells is very small in the one compared with the diminution observed in the other.

But although variations of this kind may serve to explain certain of the differences which we find in the behaviour of the gland-cells of different animals, they still leave a great deal to be explained. Why do some cells form a non-granular zone at their outer border, others at their inner border, whilst others again form no non-granular zone at all?

We may attempt to explain these differences, still having recourse to the three main changes which we have seen take place in all the cells. We may suppose that these three processes proceed at unequal rates in the outer and inner portions of the cells. Obviously a non-granular region might be formed in any part of a cell either by a more rapid using up of granules, or by a more rapid growth of protoplasm in that part.

I shall not here discuss the cause of the non-granular border which is formed in the *inner* portion of cells, such as those of the gastric glands of Frogs. For although

I think that it is due to the granules being used up in the one part of the cell faster than in the other, I am unable to offer any satisfactory proof that it is so.

On the cause of the *outer* non-granular zone which is formed in the œsophageal glands of the Frog, I would, however, offer a few remarks. I take the œsophageal glands only as a convenient example, my remarks will equally apply to the gastric glands of the Newt and other glands in which an outer non-granular zone is formed during secretion.

In the œsophageal glands we have seen that the cell-granules diminish in size during activity; if, then, in any part of the cells the granules are being used up more quickly than in another the granules will be smaller in that part than elsewhere. The diminution in the size of the granules takes place, however, equally in all parts of the cell, consequently we may conclude that the non-granular zone is not caused by any variation in the rate of using up granules in the inner and outer portions of the cells.

We have then to fall back on local variations in the rate of growth of the cell-protoplasm to explain the appearances. Will the assumption of a rapid growth of protoplasm at the outer part of the cell suffice to explain the formation of the outer non-granular zone? I think not, for several reasons. It seems to me necessary to assume that there is another factor in the result, viz.: a translation of the granules towards the lumen by the cell.

In the œsophageal and other similar glands the first sign of the subsequent zones is a diminution in the number of granules in the outer portion of the cell. If this is due to a growth of protoplasm the protoplasm must grow in the outer one-third to one-half of the cell. How, then, can the further increase of protoplasm give rise to a non-granular outer zone and a densely granular inner one at all sharply marked off the one from the other? In passing from the granular to the non-granular zone there should be a very obvious region where the granules become fewer and fewer.

In certain circumstances the œsophageal glands of the Frog contain comparatively few granules, these being scattered throughout the cells (see Plate 77, fig. 7). During activity such glands show distinct zones. How could zones in such a case be produced by a growth of protoplasm unaccompanied by a direct movement of the granules in the cells?

Again, in glands when the granules have completely disappeared, can we suppose that the protoplasm has completely regrown from the periphery?

An observation of Heidenhain's* upon the pancreas of the Rabbit is, I think, in favour of the view that the granules are moved towards the lumen. He found that when a piece of fresh pancreas was warmed to about 50° C. the granules moved outwards into the clear zone, and returned to their original position when the temperature fell. He, too, pointed out† that cells which were granular throughout, on being removed from the body gradually developed an outer non-granular zone as they died.

* Pflüger's Archiv., Bd. x., s. 563, 1875.

† *Op. cit.*, s. 559.

It may no doubt be objected that no one has seen any such movement in the living cells; but then it is to be remarked that observations on living cells have not been numerous, and that no particular cell-granule has been observed to disappear although numbers do disappear.

In the cell-protoplasm there is during secretion an increase of substance capable of reducing osmic acid.

In all the cases investigated above we have seen that the protoplasm of the "quiescent" gland-cells stains very slightly with osmic acid, whilst the protoplasm of the active cells stains more or less deeply.

This is opposed to the observations of NUSSBAUM, but NUSSBAUM, I think, directed his attention so much to the granules contained by the cells that he overlooked the cell-protoplasm.

I have not been able to satisfy myself of the meaning of this increase in staining-power of the protoplasm. Since the granules split up during activity, we might imagine that the cells do not at once cast out the whole of the substances formed, but retain a certain proportion, and that this diffused throughout the cell causes it to stain more than normally with osmic acid.

There is, however, another way of looking at the fact. Generally speaking, the formation of granules goes hand-in-hand with a diminution in the power of staining of the cell-substance. In the latter period of digestion, when the granules are increasing the cell-substance stains less and less with osmic acid. In *Triton tæniatus* during the rapid increase in the granules which takes place from the fourth to the eighth hour of digestion there is a rapid diminution in the staining power of the cell.

This would indicate that the protoplasm of the cells, in passing through those changes which result in the formation of granules, uses up the substance stainable with osmic acid. This substance might be either taken up by the cell during secretion to be further assimilated, or it might be an integral part of the cell-protoplasm.

The facts we have at present are, I think, insufficient to allow any satisfactory conclusion to be drawn on this point; but whatever the cause of it may be, it is, I think, a very general phenomena of cells to stain more deeply with osmic acid during active secretion than during rest. I have on a former occasion* pointed out that this is the case in the parotid, sub-maxillary, lachrymal, and infra-orbital glands of the Rabbit.

In parotid and sub-maxillary glands the resting cells treated with osmic acid show lightly-stained cell-substance with darker-stained granules; when the cells are similarly treated after a period of secretion the cell-substance is distinctly more deeply stained, the depth of staining increasing within certain limits with the length of time during which the cells have secreted.

In the lachrymal and infra-orbital glands the resting cells treated with osmic acid

* Proc. Royal Soc., vol. xxix., p. 377, 1879; Jour. of Physiol., vol. ii., p. 261, 1879.

show a networked appearance, from the presence of a lightly-stained protoplasmic with a darker interprotoplasmic portion. When the cells are similarly treated after a period of secretion the networked appearance disappears from the outer portion of the cells; it now is homogeneous—the protoplasm stains deeper than before. After prolonged secretion the cells stain equal and fairly darkly throughout.

In all these glands, however, the increase of substance in the cell-protoplasm capable of staining with osmic acid, although distinct, is not so marked as it is in the gastric glands.

GRÜTZNER* has independently come to a somewhat similar conclusion. He found in all the salivary and gastric glands investigated by him a difference of tint in osmic acid specimens of the resting and of the active glands. The former he found to be grey-green, the latter a dirty (*schmutzig*) brown. This, indeed, represents not unfairly the general difference in tint of the glands in the respective states when examined with not too high a power. I should prefer to call the tint of the one yellow-brown and that of the other brown-black. GRÜTZNER applied his description to the cells as a whole; the yellow-brown tint is, however, in the main due to the staining of the cell-granules, the brown-black tint in the main to the staining of the cell-protoplasm.

It will be noticed that my views on the processes taking place during secretion are opposed to the lately-advanced views of STRICKER and SPINA, and on the whole similar to those of HEIDENHAIN. The theory of STRICKER and SPINA† is based upon some interesting facts observed by them in the glands of the web and nictitating membrane of the Frog. The theory so deduced holds, they think, for all secretory glands; they attempt to show that it holds for the salivary glands.

I may be pardoned for not entering into a discussion of the question as long as the objections raised by HEIDENHAIN‡ remain unanswered. I will only remark, that in my observations on serous and mucous salivary glands, on gastric and pancreatic glands, and on the liver, I have seen nothing in favour of the theory of STRICKER and SPINA, but many things against it.

The observations, an account of which has been given in this paper, were made in the Physiological Laboratory of the University of Cambridge. I cannot too much thank Dr. FOSTER for the kindness and generosity with which he has placed at my disposal all the material and apparatus belonging to him.

* GRÜTZNER, PFLÜGER'S Arch., Bd. xx., s. 399, 1879.

† STRICKER and SPINA, Sitzb. d. k. Akad der Wissensch. z. Jena, Bd. lxxx., abt. iii., 1879.

‡ HEIDENHAIN, HERMANN'S 'Hdb. d. Physiol.,' Bd. v., Th. i., s. 414, 1880.

DESCRIPTION OF FIGURES.

Unless otherwise stated the specimens from which these drawings were made were prepared in the following manner. The tissue was placed for twenty-four hours in osmic acid 1 per cent.; removed from the osmic acid and washed for ten minutes in 50 per cent. alcohol, then placed for twenty-four hours in 75 per cent. alcohol. The sections were mounted in a mixture of one part glycerine to one part water; they undergo after mounting certain slow changes; the protoplasmic portions become darker, the mucigen portions more obvious, the outlines of the cells, nuclei and nucleoli, come more or less distinctly into view. It is almost unnecessary to point out that parts of several of the figures are drawn in outline only.

PLATE 77.

Œsophageal and Gastric Glands of *Rana temporaria* (Plate 77, figs. 1 to 11, and Plate 78, fig. 1).

Figs. 1, 2, 3. To illustrate the transition from the œsophageal to the oxyntic glands.

Fig. 1. × 360. Oblique section of end-tube of œsophageal gland. The granules are large and vary somewhat in size. The outlines of the cells are indicated; in the specimen they were only with difficulty to be made out.

Fig. 2. × 250. Gland of the intermediate region. The form of the gland resembles the form of the oxyntic rather than that of the œsophageal glands; the granules are larger than the oxyntic gland granules, but smaller than those of the œsophageal glands; they resemble closely the œsophageal gland granules.

Some characters of the resting cylindrical and sub-cubical cells are also shown here.

Fig. 3. × 370. Oxyntic gland, from latter part of anterior oxyntic region. The cells contain many granules, which are, however, much smaller than those in the cells of figs. 1 and 2.

The above three specimens were taken from a Frog (February, 1879) which had been fed with a large amount of food (0·5 grm. worm) forty-six hours previously. The last remnants of the food were still in the stomach. All the glands have slight signs of activity, especially the glands of the intermediate region, in which a small outer zone is seen.

Fig. 4. × 410. Longitudinal section of neck of oxyntic gland; the specimen illustrates also the changes which take place in the sub-cubical cells of the pyloric glands. Specimen prepared from a Frog (April, 1879) which was killed twenty-five hours after a heavy meal (four worms). The tissue was allowed to stay a month in alcohol before sections were cut, hence the distinctness of the interfibrillar substance (granules) in the mucigen portion. An earlier stage of this change in the mucigen portion of the cell can be seen in Plate 78, fig. 1.

Fig. 5. × 290. End-tubes of œsophageal gland. From Frog (September, 1879) forty-five hours after feeding with large piece of sponge. The granules are much diminished in size, and form only a narrow zone around the enlarged lumen. Several end-tubes in this specimen had lost all their granules.

Fig. 6 (*a*). × 330. End-tubes of œsophageal gland. The granules have disappeared from the cells, except at one place where a few small granules border the lumen. From Frog (September, 1879), twenty-five and a half hours after feeding with sponge. In no other Frog which I have observed did the granules disappear in so short a time. In this and the preceding figure the striation of the cells is *much* too distinct.

Fig. 6 (*b*). × 410. Same specimen as preceding after three months' stay in dilute glycerine. The striation and the very fine granules which are often seen in the clear zone are here brought out.

Fig. 7. × 350. End-tube of œsophageal gland. Frog (September, 1879) fed with sponge; after twenty-two hours sponge removed; 1·5 cub. centim. strong aqueous solution of peptone injected under skin of back; animal killed twenty-six hours after this. The granules are comparatively few, and although more numerous in the inner portion of the cells, occur also in their outer portion.

Fig. 8. × 340. End-tubes of œsophageal gland. Tissue placed in absolute alcohol for twenty-four hours, then in dilute carmine for twenty-four hours. Shows stained non-granular and unstained granular zones. Frog (September, 1879) fed with sponge thirty-five hours; sponge then vomited; animal killed ten hours after.

Fig. 9. × 410. Anterior oxyntic gland. Hungry state. The cell outlines are scarcely or not at all to be made out; the granules are distinct, numerous, and scattered throughout the cells. The nuclei and cell-substance is very slightly stained. Frog (July, 1879) killed five days after a good meal.

Fig. 10. × 580. Posterior oxyntic gland. Three hours' digestion. The granules are smaller than normal, and have disappeared from the inner border of the cells, leaving a small homogeneous zone. The nuclei and cell-substance stain darker. The nuclei are placed in the outer portion of the cells, in this specimen indeed more so than usual. The corkscrew-shaped lumen is obvious. Frog (April, 1879) several days' hunger, then fed with worm, killed in three hours.

Fig. 11. × 370. Posterior oxyntic gland. Twenty-five hours after heavy meal. Frog (April, 1879) several days' hunger, fed with four worms; killed twenty-five hours after; stomach still distended with debris of food, although a considerable amount fills out the intestine. The cells bulge out into the large lumen; the homogeneous inner zone is distinct.

PLATE 78.

Fig. 1. × 640. Oxyntic gland, middle region of stomach. Frog (September, 1879) fed with sponge forty-six hours, then with worm; killed in three hours. There is a small inner non-granular zone; the cells are much diminished in size (note the amount of magnification), the granules are small and comparatively few. The sub-cubical and cylindrical cells also show marked secretory signs.

In the actual specimens the non-granular inner zone is more distinct than in the above figures, since in the figures (owing to the method used to reproduce the drawings) the inner zone appears very finely granular instead of homogeneous; further, in the actual specimens the granules disappear more from the sides of the cells than is here represented.

GASTRIC GLANDS OF *Triton tæniatus* (Figs. 2 to 7 inclusive).

Fig. 2. × 420. Anterior oxyntic gland. Resting state. Granules conspicuous, extend throughout the cells; here and there nuclei show through them. Nuclei and cell-substance very slightly stained. Mucous cells in necks of the gland tube are shown; they are less obvious than this in the freshly mounted specimen. Newt (February, 1879), several days' hunger.

Fig. 3. × 420. Posterior oxyntic gland. Resting state. The granules are smaller than those in the anterior glands. (Newt, September, 1879.) The granules, which largely hide the nuclei in the actual specimen, are not drawn here.

Fig. 4. × 420. Posterior oxyntic gland. Four hours after feeding with worm. (Newt, September, 1879.) The granules are fewer, smaller, and vary more in size; there is a small, ill-defined outer clear zone; the nuclei and cell-substance stain darker. The cells are smaller.

Fig. 5. × 440. Oxyntic gland, latter part anterior region, six hours after feeding Newt (July, 1879), which had long fasted. The cells have become much smaller, and lost all their granules; the cell-substance stains deeply. In the most anterior glands in this specimen a few granules were left.

Fig. 6. × 325. Anterior oxyntic gland. Newt (July, 1879) fed with sponge; in eighteen hours sponge vomited; killed six hours later. The zones are sharply marked the one from the other. (2nd type, see p. 689.) The cells are nearly of normal size, the granules somewhat smaller than in the normal hungry animal. In the actual specimen they vary more in size than is here figured.

Fig. 7. × 280. Posterior oxyntic gland. Newt (December, 1879) fed with sponge forty-eight hours. The granules are small, and although they are absent from the outer region of the cells, yet do not form a distinct non-granular zone. (1st type, see p. 689.) This is an early stage of recovery. Deeply stained fat-globules are seen in groups in the outer portions of the cells (see p. 655).

Gastric Glands of Snake.

Figs. 8, 9, 10 are drawn from different regions of the same stomach, viz.: that of a Snake which was killed sixty hours after having eaten a Frog. A considerable part of the legs of the Frog were still nearly intact.

Fig. 8. × 380. Oxyntic gland from anterior region of stomach.

Fig. 9. × 380. Oxyntic gland from front part of the middle region of the stomach. Digestive changes are obvious. Only the lower part of the gland is drawn.

Fig. 10. × 420. Posterior oxyntic gland. The digestive changes are here very marked; in this, as in preceding glands, the tubes usually branch; they are not drawn here.

Fig. 11 × 280. Oxyntic gland, middle region of stomach. Hung., Snake; only the lower part is drawn. This represents fairly closely the general condition of the glands drawn in figs. 9 and 10, before digestive changes set in.

In this figure the protoplasm of the cells is made rather too dark, and most of the granules rather too small.

XVII. *Memoir on* ABEL'S *Theorem.*

By R. C. ROWE, *M.A., Fellow of Trinity College, Cambridge.*

Communicated by A. CAYLEY, *LL.D., F.R.S., Sadlerian Professor of Mathematics in the University of Cambridge.*

Received May 27,—Read June 10, 1880.

THE object of this paper is to present in a shortened and simplified form the processes and the results of ABEL'S famous memoir 'Sur une propriété générale d'une classe très-étendue de fonctions transcendantes,' composed and offered to the French Institute in 1826, but first published in the 'Mémoires des Savans Étrangers' for 1841 (pp. 176–264).

The generality and the power of this memoir are well known, but its form is not attractive. BOOLE indeed in a paper on a kindred subject (Phil. Trans. for 1857, pp. 745–803) says: "As presented in the writings of ABEL . . . the doctrine of the comparison of transcendants is repulsive, from the complexity of the formulæ in which its general conclusions are embodied." BOOLE'S theorems however escape this charge only with loss of the generality which makes ABEL'S valuable.

But this complexity is rather apparent than fundamental. It is here attempted, by re-arrangement of parts, by separation of essential from non-essential steps, by changes of notation, in particular by the introduction of a symbol and a theorem discussed by BOOLE in the paper already referred to and by the addition of examples of the processes and results, to reduce this part of an important subject to a shape more simple, while no less general, than the original.

The first of the three sections into which the following paper is divided contains (arts. 1–10) the investigation of the principal theorem of ABEL'S memoir: these articles correspond to pp. 176–196 of the original, but are much simplified by the aid of BOOLE'S proposition: the theorem is written at the end of art. 8 in the form

$$\Sigma\int f(x, y)dx=\Theta\left[\frac{1}{f_2(x)F_0(x)}\right]F_0(x)\Sigma\frac{f_1(x, y)}{\chi'(y)}\log\theta(y)+C$$

and answers to ABEL'S equation (37), p. 193.

In art. 11 three examples are given of ABEL'S theorem. Those have been chosen of which the results were well known (*e.g.*, the circular and elliptic functions) with a view to the comparison of this and less general methods.*

* For other methods of solution compare LESLIE ELLIS, B.A. Reports for 1846, p. 38; LEGENDRE, 'Fonc. Ell.,' t. iii., p. 192; BOOLE, *loc. cit.*, arts. 18, 24.

In the second section (arts. 12–20) it is shown to follow from the results of the first that the sum of any number of integrals of the form considered may be expressed in terms of a definite number of such integrals, and the question what is the least value of this definite number is discussed: the result is stated at the end of art. 20. These articles correspond to pp. 211–228 in the original; they are rendered more direct by the nomenclature of 'major terms' and 'sets,' the introduction of the letter τ, and various minor changes of notation.

Art. 21 contains an example of the method of this section.

The third section contains two distinct parts: first, a generalization (art. 22) of the theorem of Section I., showing that a similar expression to that obtained there may be found for the sum of any number of such integrals each multiplied by any rational number positive or negative, integral or fractional; secondly, an investigation (art. 23) of the conditions necessary that the algebraic expression obtained for the sum of the integrals considered in Section I.—*i.e.*, the right-hand member in the main theorem—may reduce to a constant. This article corresponds to pp. 196–208 in ABEL, but the demonstration is greatly shortened and simplified by its being placed after (instead of, with ABEL, before) Section II.

ABEL concludes by applying his methods to the case of integrals of the form

$$\int \frac{\phi}{\sqrt[m]{\psi}} dx.$$

I have succeeded in shortening the necessary work, but my process and result are so similar to those of the original as hardly to be worth reproducing here.

An appendix contains an algebraical lemma and a list—it is hoped complete—of the errata in the original memoir. It appeared to the writer worth while to attempt to save subsequent readers the considerable inconvenience these errata had caused himself.

There follows an addition from Professor CAYLEY, wherein it is shown that the expression found in art. 20 for the least value of the number of conditions connecting the variables of the integrals we sum is equal to the deficiency (*Geschlecht*) of the curve represented by the equation $\chi(x, y)=0$. That this least value is equal to the deficiency is a leading result in RIEMANN'S theory of the Abelian integrals; the assumptions made in the text as to the form of the roots of the equation $\chi(x, y)=0$ considered as an equation for the determination of y are equivalent to the assumption that the curve $\chi(x, y)=0$ has certain singularities; and it is in the addition shown that the resulting value of the deficiency, as calculated by the formulæ in Professor CAYLEY'S paper 'On the Higher Singularities of a Plane Curve,' Quart. Math. Journ., vol. vii. (1866), pp. 212–222, has in fact the foregoing value.

Section I.

1. The general question to which an answer is sought in what is called the Theory of the Comparison of Transcendants may be stated thus:—

Is it always possible to establish, between the values for different variables of the integral of an algebraic function however complex, algebraic relations: the variables themselves being connected by any requisite algebraic laws?

If, for example,

$$\int X dx = F(x)$$

where X is any algebraic function of x, rational or irrational, integral or fractional, is it necessarily possible by connecting $x_1, x_2, \ldots x_n$ by any requisite algebraic laws to obtain an algebraic (or logarithmic) expression for the sum

$$F(x_1)+F(x_2)+\ldots+F(x_n)?$$

This question is suggested on the one hand by such well-known results as

$$F(x_1)+F(x_2)=\text{constant, where } X=\frac{1}{\sqrt{1-x^2}}, \text{ if } x_1^2+x_2^2=1$$

and

$$F(x_1)+F(x_2)+F(x_3)=0 \text{ where } X=\frac{1}{\sqrt{1-x^2.1-k^2x^2}}$$

if

$$4(1-x_1^2)(1-x_2^2)(1-x_3^2)=(2-x_1^2-x_2^2-x_3^2+k^2x_1^2x_2^2x_3^2)^2,$$

and on the other hand by the possibility of finding algebraical expressions for many symmetric functions of the roots of equations though these roots may not separately be so expressible.

It is in fact this combination of the theory of integrals and the theory of equations which furnishes the key to the problem; enabling us to express the requisite algebraical laws very concisely by a *single* equation of which the variables are roots, and whose coefficients are not independent but connected by a corresponding number of relations.

2. *The expression of the function to be integrated.*

To escape the inconvenience of fractional and irrational forms we first introduce two new functions and a fresh variable.

Whatever be the nature of the function X—the subject of integration in the transcendants we are considering—it may be written

$$f(x, y)$$

where f is a rational (but not necessarily integral) function of x and y, while y is determined as a function of x from the equation

$$\chi(y)\equiv y^n+p_{n-1}y^{n-1}+p_{n-2}y^{n-2}+\ldots+p_1 y+p_0=0$$

the p's being rational integral functions of x.

This is clear since any explicit irrational function is the root of an equation with integral and rational coefficients, in which, by a suitable change of variable, the highest coefficient can be made unity.

4. The shape in which it is most convenient to deal with $f(x, y)$, and in which we shall in future assume it to be expressed, is obtained when its denominator is made the product of $\chi'(y)$—the differential coefficient of $\chi(y)$ with respect to y—and a function of x only.

This can always be done; for let

$$\begin{aligned} f(x, y)&=\frac{F_1(x, y)}{F_2(x, y)}\\ &=\frac{F_1(x, y)\chi'(y)}{F_2(x, y)\chi'(y)}\\ &=\frac{F_1(x, y)\chi'(y)F_2(x, y_2)F_2(x, y_3)\ldots F_2(x, y_n)}{\chi'(y).F_2(x, y_1)F_2(x, y_2)F_2(x, y_3)\ldots F_2(x, y_n)} \end{aligned}$$

$y_1, y_2 \ldots y_n$ being the n roots of the equation

$$\chi=0$$

and therefore functions of x; and y_1 being the root which we have before denoted by y.

Now the product $F_2(x, y_1)\ldots F_2(x, y_n)$ involving only symmetrical functions of the y's may be expressed as a function of x only; while, using the equations

$$\sum_{r=2}^{r=n} y_r = \sum_{r=1}^{r=n} y_r - y_1$$

$$\sum_{r=2}^{r=n}\sum_{s=2}^{s=n} y_r y_s = \sum_{r=1}^{r=n}\sum_{s=1}^{s=n} y_r y_s - y_1 \sum_{r=2}^{r=n} y_r$$

$$\&c. = \&c.$$

and, lastly,

$$y_2 y_3 \ldots y_n = \frac{(-)^n p_0}{y_1},$$

the product

$$F_2(x, y_2)F_2(x, y_3)\ldots F_2(x, y_n)$$

can be expressed as a rational function of x, y; while $F_1(x, y)$ and $\chi'(y)$ are rational and integral functions.

So $f(x, y)$ the subject of integration is reduced to the form

$$\frac{f_1(x, y)}{f_2(x)\chi'(y)}$$

in which it will hereafter be used.

5. *The equation whose roots are the variables of the functions we compare.*

This equation is clearly not arbitrary; for if it were we could choose it linear; and having then only a single integral, should be required to find for it an algebraic (or logarithmic) expression, a thing generally impossible.

We shall find it sufficient to take, for this equation, the result of eliminating y between χ and any other integral function of x, y, which, by the use of χ, can, of course, be made of (at most) the $(n-1)^{\text{th}}$ degree in y.

Let this second function be

$$\theta(y) \equiv q_{n-1}y^{n-1} + q_{n-2}y^{n-2} + \ldots + q_1 y + q_0$$

and let the result of elimination, viz.:—

$$\theta(y_1)\theta(y_2) \ldots \theta(y_n)$$

be denoted by E.

$\theta = 0$ may be called the *equation of condition.*

We assume $q_0, q_1 \ldots q_{n-1}$ to be rational integral functions of x; while any number of the coefficients in these functions are arbitrary: call them $a_1, a_2, \ldots$

E will then be a rational integral function of x and these quantities $a_1, a_2, \ldots$

We may then either (1) take the roots of the equation $E = 0$,—$a_1, a_2, \ldots$ being considered absolute constants—as the upper limits of our integrals (of which alone we view these integrals as functions); or (2) since by a due alteration of the a's we may produce any possible simultaneous alteration of the x's, we may consider the variables x in the different integrals as, in the passage from the lower to the higher limit, always connected by the equation $E = 0$, in which now $a_1, a_2, \ldots$ are a system of variables with which the variation of x has to be connected. The latter, as the more general and powerful hypothesis, is to be preferred.

$E = 0$ may be called the *equation of the limits*, or the *equation of transformation.*

6. It may happen that, owing to a relation connecting the a's, the equation $E = 0$ is satisfied by values of x independent of these new variables. This relation, since one of the θ's of which E is the product will vanish for this value of x and θ is linear in the a's, must be a linear relation. We will then suppose

$$E(x, a_1, a_2, \ldots) = F_0(x)F(x)$$

where $F_0(x)$ is independent of the a's; and, the degree of $F(x)$ being μ, let its roots be

$x_1, x_2, \ldots x_n$; let the corresponding values of y, the root of χ with which we are concerned, be $y_{11}, y_{12}, \ldots y_{1n}$.*

7. Having expressed $f(x, y)$ in a convenient shape we have next *to transform the* dx *of our integrals into the differentials of the new variables.*

If δ denotes the operation of differentiating with regard to our new variables we have from the equation $F=0$ by which x is connected with them

$$F'(x)dx+\delta F(x)=0$$

But

$$\delta E=F_0(x)\delta F(x)$$

therefore

$$dx=-\frac{\delta E}{F_0(x)F'(x)}$$

Again

$$E=\theta(y_1)\theta(y_2)\ldots\theta(y_n)$$

therefore

$$\delta E=\sum_{r=1}^{r=n}\frac{E}{\theta(y_r)}.\delta\theta(y_r)$$

* As an example of these processes let

$$X=\frac{1}{\sqrt{1+x^4}}$$

A natural assumption is

$$\chi(y)\equiv y^2-(1+x^4)=0$$

so that

$$f(x, y)=\frac{1}{y}$$

Take for the second function the form

$$\theta(y)\equiv y-(1+a_1x+a_2x^2)$$

and on elimination we find

$$E(x, a_1, a_2)\equiv(a_2^2-1)x^5+2a_1a_2x^3+(a_1^2+2a_2)x+2a_1=0$$

Now, if we had

$$a_1+a_2=-1+\sqrt{2}\ \text{(a } \textit{linear} \text{ relation)}$$

$E=0$ would be satisfied on making $x=1$ and we should have

$$F_0(x)=x-1$$

while

$$F(x)=(a_2^2-1)x^3+(2a_1a_2+a_2-1)x+a_1^2+2a_1a_2+2a_2^2+a_2-1$$

and may be expressed in terms of a_1 alone.

We should also have

$$f(x, y)=\frac{1}{y}=\frac{2}{2y}=\frac{2}{\chi'(y)}$$

so that

$$f_1(x, y)=2, f_2(x)=1.$$

Now (using as before y or y_1 indifferently for the root with which we are concerned), we have $\theta(y_1)=0$: whence if $\lambda(x, y)$ be any rational function

$$\lambda(x, y)\delta\mathrm{E}=\lambda(x, y)\sum_{r=1}^{r=n}\frac{\mathrm{E}}{\theta(y_r)}.\delta\theta(y_r)$$

$$=\lambda(x, y_1)\frac{\mathrm{E}}{\theta(y_1)}.\delta\theta(y_1)$$

all the other terms in Σ vanishing,

$$=\sum_{r=1}^{r=n}\lambda(x, y_r)\frac{\mathrm{E}}{\theta(y_r)}.\delta\theta(y_r)$$

if we introduce a set of vanishing terms.

We have then obtained an expression for dx and a convenient modification of the result when the differential is multiplied by any function λ of x and y.

So, finally,

$$f(x, y)dx=-\frac{f_1(x, y)}{\chi'(y)f_3(x)}.\frac{\delta\mathrm{E}}{\mathrm{F}_0(x)\mathrm{F}'(x)}$$

$$=-\frac{1}{f_3(x)\mathrm{F}_0(x)\mathrm{F}'(x)}\sum_{r=1}^{r=n}\frac{f_1(x, y_r)}{\chi'(y_r)}.\frac{\mathrm{E}}{\theta(y_r)}\delta\theta(y_r).$$

8. From this point a symbol and theorem due to Boole* furnish a short path to the result. The symbol is thus defined:—

"If $\phi(x)f(x)$ be any function of x composed of two factors $\phi(x), f(x)$, whereof $\phi(x)$ is rational, let $\Theta[\phi(x)]f(x)$ denote the result obtained by successively developing the function in ascending powers of each simple factor $x-a$ in the denominator of $\phi(x)$, taking in each development the coefficient of $\frac{1}{x-a}$, adding together the coefficients thus obtained and subtracting from the result the coefficient of $\frac{1}{x}$ in the development of the same function $\phi(x)f(x)$ in descending powers of x."†

Boole's theorem is the following:—

"If $\phi(x)$ be any rational function of x and if $\mathrm{E}=0$ be any equation, rational and integral with respect to x, by which x is connected with a new set of variables $a_1, a_2, \ldots$ then, provided that $\phi(x)$ does not become infinite when $\mathrm{E}=0$, we have

$$\Sigma\phi(x)=-\Theta[\phi(x)]\frac{d\log\mathrm{E}}{dx}"$$

the Σ' indicating summation for the various roots of the equation $\mathrm{E}=0$.

* Phil. Trans. for 1857, pp. 751, 757.

† Cauchy employs in his 'Calcul des Résidus' a symbol ξ only differing from Boole's Θ by not including the subtractive term last mentioned. Any theorem can be instantly transferred from the one notation to the other.

Assuming the truth of this theorem we may proceed with the investigation as follows:—

Since $f_1(x, y)\frac{\mathrm{E}}{\theta y}\delta\theta y$ is a rational integral function of x, y and may therefore be expressed in the form $\Sigma \mathrm{P}_r y^r$, P_r being a rational integral function of x and r a positive integer not greater than $n-1$, while y is a root of the equation $\chi(y)=0$, we have, by a known theorem of partial fractions,*

$$\Sigma\frac{f_1(x, y)\frac{\mathrm{E}}{\theta y}\delta\theta y}{\chi'(y)}=\mathrm{P}_{n-1}$$

We have then, by art. 7,

$$\Sigma f(x, y)dx=\Sigma\left\{\frac{-\mathrm{P}_{n-1}}{f_2(x)\mathrm{F}_0(x)\mathrm{F}'(x)}\right\}$$

$$=\Theta\left[\frac{\mathrm{P}_{n-1}}{f_2(x)\mathrm{F}_0(x)\mathrm{F}'(x)}\right]\frac{d\log \mathrm{F}}{dx}.$$

By Boole's theorem this

$$=\Theta\left[\frac{\mathrm{P}_{n-1}}{f_2(x)\mathrm{F}_0(x)\mathrm{F}'(x)}\right]\frac{\mathrm{F}'(x)}{\mathrm{F}(x)}$$

$$=\Theta\left[\frac{1}{f_2(x)\mathrm{F}_0(x)}\right]\frac{\mathrm{P}_{n-1}}{\mathrm{F}(x)}.$$

For since P_{n-1} is an integral function it contributes nothing to the interpretation of Θ by being within the square bracket: and, *if we assume that* $\mathrm{F}'(x)$ *and* $\mathrm{F}(x)$ *have no common factor* (which is also the case for $\mathrm{F}'(x)$—which contains the a's—and $\mathrm{F}_0(x)$ and $\mathrm{F}_2(x)$—which do not), we shall have in the expansion of $\frac{\mathrm{P}_{n-1}}{f_2(x)\mathrm{F}_0(x)\mathrm{F}'(x)}$ no term involving the reciprocal of a linear factor of $\mathrm{F}'(x)$, which therefore may also be brought out of the square bracket.

The expression last obtained

$$=\Theta\left[\frac{1}{f_2(x)\mathrm{F}_0(x)}\right]\frac{\mathrm{E}}{\mathrm{F}(x)}\Sigma\frac{f_1(x, y)}{\chi'(y)}\frac{\delta\theta y}{\theta y}$$

$$=\Theta\left[\frac{1}{f_2(x)\mathrm{F}_0(x)}\right]\mathrm{F}_0(x).\Sigma\frac{f_1(x, y)}{\chi'(y)}\frac{\delta\theta y}{\theta y}$$

Under this form the sum is immediately integrable, for the new variables (of which alone this is now a function) occur only in the factor $\frac{\delta\theta y}{\theta y}$.

Integrating we find

$$\Sigma\int f(x, y)dx=\Sigma\int\frac{f_1(x, y)}{f_2(x)\chi'(y)}dx=\Theta\left[\frac{1}{f_2(x)\mathrm{F}_0(x)}\right]\mathrm{F}_0(x)\Sigma\frac{f_1(x, y)}{\chi'(y)}\log\theta y+\mathrm{C}.$$

* See also note on art. 10, (i.).

This is the general theorem for the summation of integrals of any form of which we were led to suspect the existence.

It corresponds to that numbered (37) on page 193 in ABEL's Memoir (and which should be called "ABEL's Theorem," though that name is frequently given to the very narrow case of it discussed on page 255), while it is more concise through the introduction of the symbol Θ, and more intelligible through the absence of the letter ν.*

9. In general, as has been said, the function E has no factor independent of the a's, *i.e.*, $F_0(x)=1$.

In this case the formula of the last article takes the simpler form

$$\Sigma\int f(x,y)dx=\Theta\left[\frac{1}{f_2(x)}\right]\Sigma\frac{f_1(x,y)}{\chi'(y)}\log\theta y+C$$

As an example of the expansion of Θ suppose $f_2(x)=(x-\alpha)^m$.

We have then

$$\text{coefficient of } \frac{1}{x-\alpha} \text{ in the expansion of } \frac{1}{(x-\alpha)^m}\Sigma\frac{f_1(x,y)}{\chi'(y)}\log\theta y$$

$$\textit{i.e.}, \text{ of } \frac{1}{(x-\alpha)^m}\Gamma(x), \text{ say,}$$

$$\textit{i.e.}, \text{ of } \frac{1}{(x-\alpha)^m}\{\Gamma(\alpha)+(x-\alpha)\Gamma'(\alpha)+\ldots\}$$

$$=\frac{1}{m-1}\Gamma^{m-1}(\alpha)$$

$$\textit{i.e.}, \quad =\frac{1}{m-1}\cdot\frac{d}{d\alpha}\Big|^{m-1}\left\{\Sigma\frac{f_1(\alpha,A)}{\chi'(A)}\log\theta A\right\}$$

where A is the value of y corresponding to $x=\alpha$; and—representing by $C_{\frac{1}{x}}\lambda(x)$ the coefficient of $\frac{1}{x}$ in the descending expansion of $\lambda(x)$—

$$\Sigma\int f(x,y)dx\equiv\Sigma\int\frac{f_1(x,y)}{(x-\alpha)^m\chi'(x)}dx$$

$$=\frac{1}{m-1}\frac{d}{d\alpha}\Big|^{m-1}\left\{\Sigma\frac{f_1(\alpha,A)}{\chi'(\alpha)}\log\theta A\right\}-C_{\frac{1}{x}}\left\{\frac{f_1(x,y)}{(x-\alpha)^m\chi'(y)}\log\theta y\right\}+C,$$

which is identical with ABEL's formula (44).

10. Before proceeding to examples of the use of the general theorem one or two points in the proof and the result should be alluded to.

(i.) *A limitation to the form of the function θ.*

In choosing this function we may not make $q_1=0, q_2=0, \ldots q_{n-1}=0$ simultaneously:

* The want of clearness spoken of is due to an ambiguity in the important sentence (p. 187) in which ABEL implicitly defines the letter ν which is to appear prominently in his enunciation of the final theorem. But it is hardly necessary to dwell on a difficulty which the method of the text avoids.

in other words, our function must not reduce itself so as to contain x only. This is clear *à priori*; for if it should so reduce itself we might choose for q_0 a linear function of x, which is generally impossible (art. 5).

It will be useful to examine at what point the assumption vitiates the subsequent demonstration. We should, in fact, have

$$\begin{aligned} \mathrm{E} &= \theta(y_1)\theta(y_2)\ldots\theta(y_n) \\ &= q_0 q_0 \cdots q_0 \\ &= q_0^n \end{aligned}$$

so that

$$\frac{\mathrm{E}}{\theta(y_r)} = q_0^{n-1}$$

and this vanishes for all the values of x obtained by putting $\mathrm{E}=0$, so that the right-hand side of the equation

$$f(x,y)dx = -\frac{1}{f_2(x)\mathrm{F}_0(x)\mathrm{F}'(x)}\Sigma\frac{f_1(x,y_r)}{\chi'(y_r)}\cdot\frac{\mathrm{E}}{\theta(y_r)}\delta\theta y_r$$

is identically zero, and the whole process invalid.*

* There is one case in which the function θ may be legitimately reduced to the single term q_0; viz. the case when χ is a linear function of y.

It is plain that, as $n=1$, we have not the difficulty of repeated roots which generally vitiates the result of this assumption.

In fact, let

$$\chi(y) = y + \alpha$$

while

$$\theta = a_0 + a_1 x + \ldots + x^n$$

Then

$$\mathrm{E} = \mathrm{F}(x) = \theta$$

and, as by p. 718,

$$dx = -\frac{\delta\mathrm{F}(x)}{\mathrm{F}'(x)} = -\frac{\delta\theta}{\theta'}$$

we have

$$\Sigma y dx = \Sigma\frac{\alpha}{\theta'}\cdot da_0 + \Sigma\frac{\alpha x}{\theta'}\cdot da_1 + \ldots + \Sigma\frac{\alpha x^{n-1}}{\theta'}\cdot da_{n-1}$$

As an example of which formula, let $\alpha = x^m$

so that

$$-\frac{\Sigma x^{m+1}}{m+1} = \int\Sigma\frac{x^m}{\theta'}da_0 + \int\Sigma\frac{x^{m+1}}{\theta'}da_1 + \ldots + \int\Sigma\frac{x^{m+n-1}}{\theta'}da_{n-1}$$

Put $m=0$ and we have

$$-\Sigma x = \int\Sigma\frac{1}{\theta'}da_0 + \int\Sigma\frac{x}{\theta'}da_1 + \ldots$$

But

$$-\Sigma x = a_{n-1}$$

whence

$$\Sigma\frac{x^k}{\theta'} = 0 \text{ if } k < n-1$$

while

$$\Sigma\frac{x^{n-1}}{\theta'} = 1$$

And this is the theorem (easy to prove otherwise) which was assumed in the course of the general demonstration on page 719.

(ii.) *The assumption* (in BOOLE's theorem) *that* ϕ *is not rendered infinite by the values which satisfy the equation* $E=0$; and *the assumption* (in art. 8) *that* $F(x)$ *and* $F'(x)$ *have no common factor.*

These assumptions are identical: for ϕ is rendered infinite by the vanishing of $f_2(x)F_0(x)F'(x)$, and, since the roots of F are all functions of the a's, they cannot satisfy the equations $f_2(x)=0$, $F_0(x)=0$, into which no a enters.

If then F and F' have no common factor, the first assumption is justified.

We assert in this that $F=0$ is not an equation possessing equal roots—*i.e.*, that $x_1, x_2, \ldots x_n$ are all unequal. Suppose, on the contrary, that we have equal roots—say $x_1=x_2=x_3$.* If then y_1, y_2, y_3 are the corresponding roots of χ we shall have

$$\theta(y_1)=0, \quad \theta(y_2)=0, \quad \theta(y_3)=0$$

for the same value x_1 of x; and therefore in the expression of

$$-\frac{1}{f_2(x)F_0(x)F'(x)}\Sigma\frac{f_1(x,y_r)}{\chi'(y_r)}\cdot\frac{E}{\theta y_r}\delta\theta y_r$$

we have a term of the form $\frac{0}{0}$, viz.: that due to the root $x=x_1$, and it will be three times repeated.

We see then the character of the difficulty introduced by the equality of roots. It does not altogether vitiate the solution; it only requires that we should modify it by using, instead of the equations $\theta(y_1)=0$, $\theta(y_2)=0$, $\theta(y_3)=0$, the equations

$$\theta(y_1)=0, \quad \frac{d\theta y_1}{dx}=0, \quad \frac{d^2\theta y_1}{dx^2}=0.$$

The manner in which all the steps of the analysis and the final result are affected by this change is obvious.

11. It will now be natural to give examples of the application of the general theorem, and those are chosen the results of which are well-known, as furnishing comparison between this and other methods of research among transcendants. The second and third are treated by BOOLE, in the paper frequently referred to, as examples of his less general theorem.

I. *The function* $\sin^{-1}x$.

Let

$$X=\frac{1}{\sqrt{(1-x^2)}},$$

and take

$$\chi(y)=y^2+x^2-1,$$

* The reasoning will be applicable to any other number of equalities among the roots.

so that

$$f(x, y)=\frac{1}{y}\,;\ f_1(x, y)=2\,;\ f_2(x)=1.$$

Also let

$$\theta(y)=y+x-a.^*$$

Eliminating y we get

$$\mathrm{E}\equiv 2x^2-2ax+a^2-1=0$$

as the equation of the limits.

If x_1, x_2 are the roots of this equation we easily find

$$x_1^2+x_2^2=1.$$

The theorem then gives

$$\Sigma\int\frac{dx}{\sqrt{(1-x^2)}}=\Theta[1]\Sigma\frac{1}{y}\log\,(y-a+x)+\mathrm{C}$$

$$=-2\mathrm{C}_1\,\Sigma\frac{1}{y}\left(\log x+\frac{y-a}{x}-\frac{(y-a)^2}{2x^2}+\ldots\right)+\mathrm{C}.$$

But $\Sigma\frac{1}{y}=0$, wherefore the right-hand side reduces to a constant, and we have the result that

$$\int_0^{x_1}\frac{dx}{\sqrt{(1-x^2)}}+\int_0^{x_2}\frac{dx}{\sqrt{(1-x^2)}}$$

is constant if

$$x_1^2+x_2^2=1,$$

and so

$$=\int_0^1\frac{dx}{\sqrt{(1-x^2)}}\,;$$

and this is, of course, the well-known theorem that $\theta+\phi=\frac{\pi}{2}$ if $\sin^2\theta+\sin^2\phi=1$, (the angles being restricted to the first quadrant).

II. *The elliptic functions.*

As a second example take

$$\mathrm{X}=\frac{a+bx^2}{(1+nx^2)\sqrt{(1-x^2)(1-c^2x^2)}}$$

and let

$$\chi(y)\equiv y^2-(1-x^2)(1-c^2x^2)$$

so that

$$\int\mathrm{X}dx=\int\frac{2(a+bx^2)}{(1+nx^2)\chi'(y)}dx$$

and

$$f_1(x, y)=2(a+bx^2)$$

$$f_2(x)=1+nx^2.$$

* To choose the more general form $y+bx-a$ leads by similar steps to a less interesting result.

Also take

$$\theta(y) \equiv y-(1+px+qx^2),$$

so that removing the factor $x=0$ (see p. 732), we get by elimination of y between χ and θ

$$\mathrm{E} \equiv (q^2-c^2)x^3+2pqx^2+(p^2+2q+c^2+1)x+2p=0.$$

It is clear that, in general, no linear relation connects the coefficients of this equation, so that $\mathrm{F}_0(x)=1$; and the formula reduces to

$$\Sigma\int \mathrm{X}dx = \Theta\left[\frac{1}{f_3(x)}\right]\Sigma\frac{f_1(x,\,y)}{\chi'(y)}\log\theta y$$

$$= \Theta\left[\frac{1}{1+nx^2}\right]\Sigma\frac{a+bx^2}{y}\log\{y-(1+px+qx^2)\}$$

$$= \frac{1}{n}\Theta\left[\frac{1}{\left(x+\frac{\iota}{\sqrt{n}}\right)\left(x-\frac{\iota}{\sqrt{n}}\right)}\right]\Sigma\frac{a+bx^2}{y}\log\{y-(1+px+qx^2)\}$$

where, as usual, $\iota=\sqrt{-1}$.

Therefore

$$n\Sigma\int \mathrm{X}dx = -\frac{\sqrt{n}}{2\iota}\Sigma\frac{a-\frac{b}{n}}{y}\log\left\{y-\left(1-\frac{p\iota}{\sqrt{n}}-\frac{q}{n}\right)\right\}$$

$$+\frac{\sqrt{n}}{2\iota}\Sigma\frac{a-\frac{b}{n}}{y}\log\left\{y-\left(1+\frac{p\iota}{\sqrt{n}}-\frac{q}{n}\right)\right\}$$

$$-n\mathrm{C}_x^1\frac{a+bx^2}{1+nx^2}\Sigma\frac{\log\{y-(1+px+qx^2)\}}{y}$$

Now the last term in general vanishes.

For $$\Sigma\frac{\log\{y-(1+px+qx^2)\}}{y} = \Sigma\frac{\log(-qx^2)}{y}+\Sigma\frac{\log\left(1+\frac{px+1-y}{qx^2}\right)}{y}$$

and $$\Sigma\frac{1}{y}=0$$

while C_x^1 in the term $\Sigma\dfrac{\log\left(1+\frac{px+1-y}{qx^2}\right)}{y}$ is $\Sigma\dfrac{p}{qy}$

and this vanishes.

Therefore the first term of the descending expansion of Σ involves x^{-2}, while that of $\dfrac{a+bx^2}{1+nx^2}$ begins with x^0;

wherefore $$\mathrm{C}_x^1\frac{a+bx^2}{1+nx^2}\Sigma\frac{\log\{y-(1+px+qx^2\}}{y}=0.*$$

* There is an exceptional case if $\left.\begin{matrix}n=0\\ b\neq 0\end{matrix}\right\}$, for then the expansion of $\dfrac{a+bx^2}{1+nx^2}$ begins with x^2; and the C_x^1 is not necessarily zero.

Next, the two values of y under the sign Σ being

$$y=\pm\sqrt{\left\{\left(1+\frac{1}{n}\right)\left(1+\frac{c^2}{n}\right)\right\}}, \text{ or say } \pm\frac{k}{n},$$

the first two terms in the expression of $n\Sigma\int X dx$ compound into

$$-\frac{\sqrt{n}}{2i}\frac{na-b}{k}\log\frac{\left\{\frac{k}{n}-\left(1-\frac{pi}{\sqrt{n}}-\frac{q}{n}\right)\right\}\left\{-\frac{k}{n}-\left(1+\frac{pi}{\sqrt{n}}-\frac{q}{n}\right)\right\}}{\left\{-\frac{k}{n}-\left(1-\frac{pi}{\sqrt{n}}-\frac{q}{n}\right)\right\}\left\{\frac{k}{n}-\left(1+\frac{pi}{\sqrt{n}}-\frac{q}{n}\right)\right\}}$$

$$=\frac{i\sqrt{n}}{2}\frac{na-b}{k}\log\frac{(n-q)^2-(k+p\sqrt{n}i)^2}{(n-q)^2-(k-p\sqrt{n}i)^2}$$

which is easily rationalised, and gives

$$\sqrt{n}\frac{na-b}{k}\tan^{-1}\frac{2pk\sqrt{n}}{(n-q)^2+p^2n-k^2}$$

which, substituting for k^2 its value $(1+n)(c^2+n)$

$$=\sqrt{n}\frac{na-b}{k}\tan^{-1}\frac{\frac{2p}{q^2-c^2}k\sqrt{n}}{1+n\left\{1+\frac{p^2-(q+1)^2}{q^2-c^2}\right\}}$$

Now, if x_1, x_2, x_3 be the roots of the equation $E=0$, we get at once the relations

$$x_1x_2x_3=-\frac{2p}{q^2-c^2},$$

$$x_1^2+x_2^2+x_3^2-2-c^2x_1^2x_2^2x_3^2=2\frac{p^2-(q+1)^2}{q^2-c^2},$$

$$(1-x_1^2)(1-x_2^2)(1-x_3^2=\left\{\frac{p^2-(q+1)^2}{q^2-c^2}\right\}^2.$$

We have then finally the following theorem.

$$\text{If} \qquad \psi(x)=\int_0 \frac{a+bx^2}{(1+nx^2)\sqrt{\{(1-x^2)(1-c^2x^2)\}}}\,dx$$

then, provided that x_1, x_2, x_3 are connected by the single relation

$$(2-x_1^2-x_2^2-x_3^2+c^2x_1^2x_2^2x_3^2)^2=4(1-x_1^2)(1-x_2^2)(1-x_3^2),$$

we have

$$\psi(x_1)+\psi(x_2)+\psi(x_3)=\sqrt{\frac{n}{(n+1)(n+c^2)}}\left(a-\frac{b}{n}\right)\tan^{-1}\frac{-\sqrt{\{n(n+1)(n+c^2)\}}x_1x_2x_3}{1+n\{1\pm\sqrt{(1-x_1^2)(1-x_2^2)(1-x_3^2)}\}}.$$ *

If we write $\sin\theta$ for x we have the corresponding expression

$$\sqrt{\frac{n}{(n+1)(n+c^2}}\left(a-\frac{b}{n}\right)\tan^{-1}\frac{-\sqrt{\{n(n+1)(n+c^2)\}}\sin\theta_1\sin\theta_2\sin\theta_3}{1+n(1\pm\cos\theta_1\cos\theta_2\cos\theta_3)}$$

for the sum of three integrals of the form

$$\int_0 \frac{a+b\sin^2\theta}{(1+n\sin^2\theta)\sqrt{(1-c^2\sin^2\theta)}}\,d\theta$$

whose variables are connected by the relation

$$(1-\cos^2\theta_1-\cos^2\theta_2-\cos^2\theta_3-c^2\sin^2\theta_1\sin^2\theta_2\sin^2\theta_3)^2=4\cos^2\theta_1\cos^2\theta_2\cos^2\theta_3\text{†}.$$

From the formula just proved we can deduce without difficulty the well-known theorems connecting the elliptic functions of each order whose variables are connected by the equation

$$1-\cos^2\theta_1-\cos^2\theta_2-\cos^2\theta_3-c^2\sin^2\theta_1\sin^2\theta_2\sin^2\theta_3+2\cos\theta_1\cos\theta_2\cos\theta_3=0\text{‡}$$

which is only another form of the familiar relation

$$\cos\theta_1=\cos\theta_2\cos\theta_3\pm\sin\theta_2\sin\theta_3\,\Delta\,\theta_1.$$

* It is here assumed that $n(n+1)(n+c^2)$ is positive. If this is not the case the imaginary $\tan^{-1}$ is replaced by a real logarithm.

† The exceptional case $\left.\begin{matrix}n=0\\ b\neq 0\end{matrix}\right\}$ in which there will be an additional term due to $C\frac{1}{2}$ must not be forgotten.

‡ We take the negative sign in the ambiguity.

For the first kind.

Here we put

$$a=1,\ b=0,\ n=0.$$

This does not fall under the exceptional case; and our formula gives

$$\mathrm{F}(\theta_1)+\mathrm{F}(\theta_2)+\mathrm{F}(\theta_3)=0.$$

For the second kind.

Here we put

$$a=1,\ b=-c^2,\ n=0.$$

This gives rise to the exceptional case.

The right-hand side of the formula vanishes. It remains to find the value of

$$-\mathrm{C}_{\frac{1}{x}}\Sigma\frac{1-c^2x^2}{y}\log\theta y$$

$$=-\mathrm{C}_{\frac{1}{x}}\frac{1-c^2x^2}{y_1}\log\frac{1+px+qx^2-y_1}{1+px+qx^2-y_1}$$

where $y_1=\sqrt{(1-x^2)(1-c^2x^2)}$

$$=2\mathrm{C}_{\frac{1}{x}}(1-c^2x^2\left\{\frac{1}{1+px+qx^2}+\tfrac{1}{3}\frac{y_1^2}{(1+px+qx^2)^3}+\ldots\right\}$$

which, clearly,

$$=-2c^2\mathrm{C}_{\frac{1}{x}}x^2\left\{\frac{1-\frac{p}{qx}+\ldots}{qx^2}+\tfrac{1}{3}\frac{(c^2x^4-\ldots)\left(1-\frac{3p}{qx}\ldots\right)}{q^3x^6}+\ldots\right\}$$

$$=2c^2\left(\frac{p}{q^3}+\frac{pc^2}{q^4}+\frac{pc^4}{q^5}+\ldots\right)$$

$$=\frac{2pc^2}{q^3-c^2}$$

$$=-c^2\sin\theta_1\sin\theta_2\sin\theta_3.$$

Therefore $\mathrm{E}(\theta_1)+\mathrm{E}(\theta_2)+\mathrm{E}(\theta_3)=-c^2\sin\theta_1\sin\theta_2\sin\theta_3.$

For the third kind.

We have to write $a=1$, $b=0$, and get

$$\Pi(n,\theta_1)+\Pi(n,\theta_2)+\Pi(n,\theta_3)$$

$$=-\sqrt{\frac{n}{(n+1)(n+c^2)}}\tan^{-1}\frac{\sqrt{n(n+1)(n+c^2)}\sin\theta_1\sin\theta_2\sin\theta_3}{1+n(1-\cos\theta_1\cos\theta_2\cos\theta_3)}$$

(or the corresponding logarithmic expression if $n(n+1)(n+c^2)$ is negative).*

* Cayley, 'Elliptic Functions,' art. 132.

III. "ABEL'S *Theorem*."

As a third example, consider a problem analogous to that of BOOLE, art. 20; but more easily reduced by ABEL'S theorem than by his.

Let

$$X \equiv \frac{\phi(x)}{\{\psi(x)\}^{\frac{m}{n}}}$$

where $\phi(x)$ is a rational integral or fractional function, $\psi(x)$ is a rational integral function, while m and n are positive integers prime to one another.

To this form any expression containing only a single term can be reduced.

Let

$$y = x^n \psi^m$$

while

$$\theta = \lambda_2 y - \lambda_1$$

λ_1 and λ_2 being rational integral functions: also let

$$\phi(x) \equiv \frac{\phi_1(x)}{\phi_2(x)}.$$

Then, eliminating,

$$E \equiv \lambda_1^n - \lambda_2^n \psi^m,$$

and, in general,

$$F_0(x) = 1.$$

So

$$X = \frac{\phi_1(x)}{\phi_2(x) y} = \frac{n y^{n-2} \phi_1(x)}{\phi_2(x) \chi'(y)}$$

so that

$$f_1(x, y) = n y^{n-2} \phi_1(x),$$
$$f_2(x) \quad = \phi_2(x).$$

Therefore

$$\Sigma \int X dx = \Theta\left[\frac{1}{\phi_2(x)}\right] \Sigma \frac{\phi_1(x)}{y} \log(\lambda_2 y - \lambda_1) + C$$

$$= \Theta[\phi(x)] \Sigma \frac{1}{y} \log(\lambda_2 y - \lambda_1) + C$$

But, if $1, \omega_1, \omega_2, \ldots \omega_{n-1}$ are the n^{th} roots of unity the values of y are

$$\psi^{\frac{m}{n}}, \ \psi^{\frac{m}{n}} \omega_1, \ldots \psi^{\frac{m}{n}} \omega_{n-1}.$$

So the last expression becomes (putting ω_0 for 1)

$$\Theta[\phi(x)] \psi^{-\frac{m}{n}} \left\{ \sum_0^{n-1} \omega \log \lambda_2 + \sum_0^{n-1} \omega \log\left(\omega \psi^{\frac{m}{n}} - \frac{\lambda_1}{\lambda_2}\right) \right\} + C$$

$$= \Theta[\phi(x)] \psi^{-\frac{m}{n}} \left\{ \sum_0^{n-1} \omega \log\left(\omega \psi^{\frac{m}{n}} - \frac{\lambda_1}{\lambda_2}\right) \right\} + C$$

since

$$\sum_0^{n-1} \omega = 0.$$

As a particular case of this result what is often called ABEL's Theorem may be adduced.

Let

$$X=\frac{f(x)}{(x-a)\sqrt{\phi_1(x)\phi_2(x)}}.$$

We have to write in the previous work

for	$\phi(x)$	$\frac{f(x)}{x-a}$
for	$\psi(x)$	$\phi_1(x)\phi_2(x)$
for	$\frac{m}{n}$	$\frac{1}{2}$

The right-hand side becomes

$$\Theta\left[\frac{f(x)}{x-a}\right]\frac{1}{\sqrt{\phi_1(x)\phi_2(x)}}\sum_0^1\omega\log\left(\omega\sqrt{\phi_1(x)\phi_2(x)}-\frac{\lambda_1}{\lambda_2}\right)+C.$$

The two values of ω are $+1$, -1.

Therefore the above

$$=\Theta\left[\frac{f(x)}{x-a}\right]\frac{1}{\sqrt{\phi_1(x)\phi_2(x)}}\log\frac{\lambda_1-\lambda_2\sqrt{\phi_1(x)\phi_2(x)}}{\lambda_1+\lambda_2\sqrt{\phi_1(x)\phi_2(x)}}.$$

This assumes a more symmetrical shape if, with ABEL, we write, not $\sqrt{\phi_1(x)\phi_2(x)}=\frac{\lambda_1}{\lambda_2}$, but $\sqrt{\frac{\phi_2(x)}{\phi_1(x)}}=\frac{\lambda_1}{\lambda_2}$; so that $\sqrt{\phi_1(x)\phi_2(x)}=\frac{\lambda_1\phi_1(x)}{\lambda_2}$.

With this alteration we get

$$\Sigma\int\frac{f(x)dx}{(x-a)\sqrt{\phi_1(x)\phi_2(x)}}=\Theta\left[\frac{f(x)}{x-a}\right]\frac{1}{\sqrt{\phi_1(x)\phi_2(x)}}\log\frac{\lambda_1\sqrt{\phi_1(x)}-\lambda_2\sqrt{\phi_2(x)}}{\lambda_1\sqrt{\phi_1(x)}+\lambda_2\sqrt{\phi_2(x)}}+C$$

$$=\frac{f(a)}{\sqrt{\phi_1(a)\phi_2(a)}}\log\frac{\lambda_1(a)\sqrt{\phi_1(a)}-\lambda_2(a)\sqrt{\phi_2(a)}}{\lambda_1(a)\sqrt{\phi_1(a)}+\lambda_2(a)\sqrt{\phi_2(a)}}$$

$$-C\frac{f(x)}{2(x-a)\sqrt{\phi_1(x)\phi_2(x)}}\log\frac{\lambda_1(x)\sqrt{\phi_1(x)}-\lambda_2(x)\sqrt{\phi_2(x)}}{\lambda_1(x)\sqrt{\phi_1(x)}+\lambda_2(x)\sqrt{\phi_2(x)}}+C$$

which is the well-known theorem referred to.

We see it to be only a particular case of a particular case of the theorem called in this paper ABEL's Theorem.

SECTION II.

12. The expression (in a form algebraic or logarithmic) of the sum $\Sigma\int X dx$ having been shown to exist, and having in fact been found, ABEL proceeds, in his art. 5, to investigate the condition that this expression should become a constant. Of the possibility of this we have been assured by the result of the first example and of the first case of the second example of art. 11. This investigation, as subordinate to the main purpose, may be conveniently postponed to the second principal inquiry with which the memoir is concerned.

This inquiry presents itself in two forms.

I. Mention was made at the outset of the "requisite algebraical laws" which connect the variables when the summation desired can be effected. And in the case of the elliptic functions we have found that in order to express the sum of three functions it is requisite that the variables should be connected by a *single* relation. We are naturally led to investigate the number of relations necessary for the same effect in the case of more complicated forms. This number, it must be said, depends not at all on the *number* of the functions we consider but only on their *form*.

II. We may also consider the matter thus:—

Representing by $\psi(x)$ the integral $\int X dx$, we have shown how to express, by the use of an operative symbol Θ, the sum

$$\psi(x_1)+\psi(x_2)+\ldots+\psi(x_\mu)$$

where $x_1, x_2, \ldots x_\mu$ are the roots of an equation

$$F(x)=0.$$

Now this equation involves a number, α, of arbitrary quantities $a_1, a_2, a_3 \ldots$

Its μ roots are functions of these α quantities. We can then find expressions for $a_1, a_2, \ldots$, in terms of α of these roots, say $x_1, x_2, \ldots x_\alpha$; and substituting these expressions in those which give $x_{\alpha+1} \ldots x_\mu$ shall have these $\mu-\alpha$ roots determined as functions of the other α.*

The result then is an expression for the sum of a series of functions

$$\psi(x_1)+\ldots+\psi(x_\alpha),$$

* This is most conveniently effected by

(1) solving for $a_1, a_2, \ldots$ the α equations—linear in a's—

$$\theta(y_1)=0, \quad \theta(y_2)=0 \ldots, \quad \theta(y_\alpha)=0,$$

where the equation $\theta(y_1)$ is the factor of E which supplies the factor $x-x_1$ to $F(x)$, and

(2) substituting the values so obtained in $F(x)$, which then becomes divisible by

$$(x-x_1)(x-x_2)\ldots(x-x_\alpha),$$

and gives as quotient an equation of the degree $\mu-\alpha$ whose coefficients are rational integral functions of (x_1, y_1), &c., and whose roots are the quantities $x_{\alpha+1}, x_{\alpha+2}, \ldots x_\mu$ which it is required to determine.

$x_1 \ldots x_n$ being any quantities whatever, in terms of an algebraic function and a number of functions of the same form whose variables are themselves definite functions of the quantities $x_1, x_2, \ldots x_n$.

The question then arises, What is the smallest number of functions in terms of which the sum may be expressed? and can the sum of *any* number of functions be expressed in terms of this smallest number?*

13. *Required the least value of which the difference between the number of roots possessed by the 'equation of the limits' and the number of constants introduced by the 'equation of condition' is susceptible.*

This difference is expressed by $\mu-\alpha$. We must put each term under a different form.

(i.) For α.

Let us express the index of the highest power of x in a function $J(x)$, supposed rational and integral, by the symbol $\overline{J(x)}$.

Then in general the number of coefficients in $J(x)$ is $\overline{J(x)}+1$: and as in θ one coefficient may without loss of generality be written unity

$$\begin{aligned}\alpha &= \text{number of coefficients in } \theta(\equiv q_{n-1}y^{n-1}+\ldots+q_0)\\ &= \Sigma\overline{q}+n-1.\end{aligned}$$

Two corrections must be introduced.

For the existence of each linear factor of F_0 implies a linear relation between the a's, and diminishes the independent number by unity. We have on this account to subtract $\overline{F_0}$. It may happen, however, that the particular form of the function renders the number of necessary relations less. Write then $\overline{F_0}-A$ as the quantity to be subtracted.

Suppose again that some of the constants are so chosen as to reduce the degree of E.†

In general μ and α are thus equally reduced; but it may happen that the form of the function renders necessary a less number of conditions. If this lessens μ by a number greater by B than the lessening of α we have to use instead of $\overline{F_0}-A$, $\overline{F_0}-A-B$.

We will however for the present drop the A and B, which would appear without alteration throughout the process, and replace them at its conclusion in the shape of a correction to the result.

* In an earlier memoir (ABEL'S works, vol. ii., xi.), this question is dismissed with the remark "il n'est pas difficile de se convaincre que, quelque soit le nombre μ on peut toujours faire en sorte que $n-\mu$ devienne independant de μ." Here the actual value of this constant is investigated.

† For example, in the case of p. 725, we put $\sqrt{1-x^3}.\sqrt{1-c^3x^3}=1+px+qx^3$, and the assumption of unity as the first term on the right reduced the resulting equation from a quartic to a cubic.

We have then

$$\alpha=\Sigma\overline{q}+n-1-\overline{\mathrm{F}_0}.$$

(ii.) *For* μ.

Since

$$\theta(y_1)\theta(y_2)\ldots\theta(y_n)=\mathrm{F}_0\mathrm{F}$$

it follows that

$$\Sigma\overline{\theta(y)}=\overline{\mathrm{F}_0}+\overline{\mathrm{F}}=\overline{\mathrm{F}_0}+\mu$$

So

$$\mu-\alpha=\Sigma\overline{\theta(y)}-\Sigma\overline{q}-n+1.*$$

Now

$$\begin{aligned}\overline{\theta(y)}&\geq\overline{q_r y^r}\\&=\overline{q_r}+r\overline{y}\end{aligned}$$

and it becomes necessary to find $\overline{y_1}, \overline{y_2}, \ldots \overline{y_n}$.

14. We require the following Lemma.

The quantities $\overline{y_1}, \overline{y_2}, \ldots \overline{y_n}$, *are in general equal in sets.*

For let $y_1=\frac{m_1}{\mu_1}$; this being a fraction in its lowest terms (and we will take the denominator positive).

Then one root of χ being, when expanded in descending powers of x,

$$y=\mathrm{A}x^{\frac{m_1}{\mu_1}}+\ldots$$

the expressions

$$\begin{aligned}y&=\mathrm{A}\omega_1 x^{\frac{m_1}{\mu_1}}+\ldots\\y&=\mathrm{A}\omega_2 x^{\frac{m_1}{\mu_1}}+\ldots\\y&=\&\mathrm{c}.\end{aligned}$$

(where $1, \omega_1, \omega_2 \ldots$ are the μ^{th} roots of unity) are also roots, and if these are $y_2, y_3 \ldots$ we have $\overline{y_1}=\overline{y_2}=\ldots$, the number equated being clearly a multiple of μ_1. Let it be $n_1\mu_1$; and write

$$\begin{aligned}\overline{y_1}=\overline{y_2}=\ldots=\overline{y_{k_1}}&\quad\text{where}\quad k_1=n_1\mu_1\\\overline{y_{k_1+1}}=\ldots=\overline{y_{k_2}}&\quad\text{where}\quad k_2-k_1=n_2\mu_2\\\&\mathrm{c}.=\&\mathrm{c}.&\\\overline{y_{k_{l-1}+1}}=\ldots=\overline{y_{k_l}}&\quad\text{where}\quad k_l-k_{l-1}=n_l\mu_l\\&\quad\text{and}\quad k_l=n\dagger\end{aligned}$$

* Here $\overline{\theta(y)}$ means the degree of $\theta(y)$ when rendered a function of x by substitution for y from the equation $\chi(y)=0$.

† This lemma is the second of the theorems laid down by ABEL in his important memoir "Sur la résolution algébrique des équations," of which consists the last article (it was never finished) in the second volume of his works.

Also let us write, for shortness,

$$\frac{m_1}{\mu_1}=\sigma_1,\ \frac{m_2}{\mu_2}=\sigma_2,\ \ldots\ \frac{m_l}{\mu_l}=\sigma_l;$$

and let these be in descending order of magnitude, so that

$$\sigma_1>\sigma_2>\sigma_3\ \ldots\ >\sigma_l.$$

We have then n_1 sub-sets, each of μ_1 terms, with index $\frac{m_1}{\mu_1}$, n_2 sub-sets each of μ_2 terms, with index $\frac{m_2}{\mu_2}$, and so on.

These quantities m_1, μ_1, m_2, μ_2, &c., can be speedily determined when χ is given, by NEWTON'S method.

Thus, write Ax^σ for y in the equation, and determine σ by the condition that in the resulting function of x the indices in two or more terms may be equal and greater than in any other term (while the condition that the sum of these terms shall vanish will determine A).*

These conditions are obviously necessary for the existence of a root $y=Ax^\sigma+\ldots$: and it is easy to prove directly that we can thus determine values of the quantities σ unique, and in descending order.

For suppose the indices after substitution to be $n\sigma$; $(n-1)\sigma+\alpha_1$; $(n-2)\sigma+\alpha_2$; ...

Then putting

$$n\sigma=(n-k)\sigma+a_k$$

we have

$$\sigma=\frac{a_k}{k};$$

* As an example, suppose that y is determined by the cubic

$$\chi\equiv y^5+p_2y^3+p_1y+p_0=0$$

while

$$\overline{p_2}=1;\ \overline{p_1}=3;\ \overline{p_0}=2.$$

Writing Ax^σ for y the exponents are

$$3\sigma,\ 2\sigma+1,\ \sigma+3,\ 2.$$

It is clear that the conditions are satisfied by making $3\sigma=\sigma+3$, *i.e.*, $\sigma=\frac{3}{2}$, while a quadratic is obtained for A, so that there are two corresponding terms and $\overline{y_1}=\overline{y_2}$.

They are also satisfied by making $\sigma+3=2$, *i.e.*, $\sigma=-1$, and a simple equation is obtained for A.

We have, then,

$$\frac{m_1}{\mu_1}\equiv\sigma_1=\tfrac{3}{2};\quad n_1=1,$$

$$\frac{m_2}{\mu_2}\equiv\sigma_2=-1;\ n_2=1.$$

and if we choose k so that $\frac{a_k}{k}$ is the greatest of the series $\frac{a_1}{1}, \frac{a_2}{2} \ldots$, we have, determined as a unique value, what we will provisionally call σ_1.

Next put

$$(n-k)\sigma+a_k=(n-s)\sigma+a_s$$

whence

$$\sigma=\frac{a_s-a_k}{s-k}$$

Now this value is to make

$$(n-k)\sigma+a_k>(n-t)\sigma+a_t$$

or

$$(t-k)a_s-(s-k)a_t>(t-s)a_k,$$

and since by interchanging s and t we get the contradictory of this inequality, it is impossible that by putting

$$(n-k)\sigma+a_k=(n-t)\sigma+a_t$$

each of these could be made $>(n-s)\sigma+a_s$.

Therefore the second step is also unique; and

$$\frac{a_s-a_k}{s-k}<\frac{a_k}{k} \quad \text{since } \frac{a_s}{s}<\frac{a_k}{k},$$

so that the second σ is less than the first and may be called σ_2.

Now, resuming the process of art. 13, divide the terms of the expression

$$\theta(y)\equiv q_{n-1}y^{n-1}+q_{n-2}y^{n-2}+\ldots+q_1y+q_0$$

into sets: calling the first k_1 of them the *first set*, the next k_2-k_1 the *second set*, and so on, the last k_l-k_{l-1} constituting the l^{th} set.

Also call that term of the first set in which when y_1 is written for y the highest resulting index of x is the largest the *major term of the first set*, call that term of the second set in which on the substitution of y_2 the same happens the *major term of the second set*, and so on.

Then I proceed

(i) to show that by a proper choice of the quantities $\overline{q_{n-1}}, \overline{q_{n-2}}, \ldots \overline{q_1}, \overline{q_0}$, which are at our disposal, we can make the major term of the first set an *absolute* major (for the substitution y_1), *i.e.*, furnish a higher index of x than is furnished by *any* other term; the major term of the second set an absolute major (for the substitution y_2), and so on,

(ii) to show that the condition of (i) is necessary in order that $\mu-\alpha$ may have the smallest value of which it is susceptible.

(iii) to find this value.

The proof of (i) is most simply conducted by successively investigating the conditions

(a) that the major term of any (say the r^{th}) set should furnish a higher power of x (for the substitution y_r) than any other *major term* furnishes,

(b) that this major term should furnish a higher power than any other not-major term furnishes.

In investigating (b) the conditions of (a) are to be supposed to hold. It will only be found necessary to supply to them a slight additional restriction in order to satisfy (b).

17. The condition for (a) is that whatever values (of course lying between 0 and $n-1$ inclusive) are given to r and s we should have

$$q_{\rho_r}+\rho_r\sigma_r > q_{\rho_s}+\rho_s\sigma_r,$$

where we have taken $q_{\rho_r}\, y^{\rho_r}$ to be the major term of the r^{th} set.

We will write this, for brevity, in the form

$$[\rho_r]+\rho_r\sigma_r > [\rho_s]+\rho_s\sigma_r \quad \ldots\ldots\ldots \quad (A)$$

so that

$$[\rho_r] \equiv q_{\rho_r}$$

If we make successively the substitutions

$$\left.\begin{matrix} r=m+1 \\ s=m \end{matrix}\right\} \qquad \left.\begin{matrix} r=m \\ s=m+1 \end{matrix}\right\}$$

we find that the above inequality requires the following :—

$$\begin{matrix} [\rho_{m+1}]-[\rho_m] > (\rho_m-\rho_{m+1})\sigma_{m+1} \\ < (\rho_m-\rho_{m+1})\sigma_m \end{matrix}$$

If then we write

$$[\rho_{m+1}]-[\rho_m] = (\rho_m-\rho_{m+1})\tau_m$$

we have

$$\begin{matrix} \tau_m > \sigma_{m+1} \\ < \sigma_m \end{matrix}$$

If we use also for $\rho_m-\rho_{m+1}$ the abbreviation $\delta\rho_m$ we have

$$[\rho_{m+1}]-[\rho_m] = \delta\rho_m.\tau_m$$

and it follows that

$$\begin{aligned} [\rho_m] &= [\rho_1]+\delta\rho_1.\tau_1+\ldots+\delta\rho_{m-1}.\tau_{m-1} \\ &= [\rho_1]+\sum_{i=1}^{i=m-1}\delta\rho_i.\tau_i \quad \ldots\ldots\ldots\ldots\ldots \quad (B) \end{aligned}$$

The condition expressed by this equation is then necessary if the inequality (A) is to hold.

It is also sufficient, as is easily shown.

For from it we obtain

if $r>s$.

$$\begin{aligned}[\rho_r]-[\rho_s]&=\delta\rho_r.\tau_s+\delta\rho_{s+1}.\tau_{s+1}+\ldots+\delta\rho_{r-1}.\tau_{r-1}\\&>(\delta\rho_s+\delta\rho_{s+1}+\ldots+\delta\rho_{r-1})\tau_{r-1}\\&>(\rho_s-\rho_r)\tau_{r-1}\\&>(\rho_s-\rho_r)\sigma_r\end{aligned}$$

if $r<s$

$$\begin{aligned}[\rho_r]-[\rho_s]&=-\delta\rho_r.\tau_r-\delta\rho_{r+1}.\tau_{r+1}-\ldots-\delta\rho_{s-1}.\tau_{s-1}\\&>-(\delta\rho_r+\delta\rho_{r+1}+\ldots+\delta\rho_{s-1})\tau_r\\&>(\rho_s-\rho_r)\sigma_r.\end{aligned}$$

The relation (B) (in which $[\rho_1]$ is entirely arbitrary and the τ's are only subject to the necessity of lying between consecutive σ's) expresses the necessary and sufficient condition for the satisfaction of (a).

18. Let us next examine (b).

The condition is expressed by the inequality

$$[\rho_m]+\rho_m\sigma_m>[\alpha]+\alpha\sigma_m$$

where α is any term of the series $0, 1, \ldots (n-1)$ which is not one of the ρ's.

Let α belong to the λ^{th} set so that

$$[\alpha]+\alpha\sigma_\lambda<[\rho_\lambda]+\rho_\lambda\sigma_\lambda$$

and let

$$[\alpha]+\alpha\sigma_\lambda=[\rho_\lambda]+\rho_\lambda\sigma_\lambda-A_\alpha \quad \ldots\ldots\ldots \quad (C)$$

A_α being a positive quantity.

We have then to make

$$[\rho_m]+\rho_m\sigma_m>[\rho_\lambda]+\rho_\lambda\sigma_\lambda+\alpha(\sigma_m-\sigma_\lambda)-A_\alpha \quad \ldots\ldots\ldots \quad (D)$$

Now this inequality clearly holds when $m=\lambda$. Again it holds when $m=\lambda+1$ provided that

$$[\rho_{\lambda+1}]-[\rho_\lambda]>-\rho_{\lambda+1}\sigma_{\lambda+1}+\rho_\lambda\sigma_\lambda+\alpha(\sigma_{\lambda+1}-\sigma_\lambda)-A_\alpha$$

i.e., if

$$(\rho_\lambda-\rho_{\lambda+1})\tau_\lambda>-\rho_{\lambda+1}\sigma_{\lambda+1}+\rho_\lambda\sigma_\lambda+\alpha(\sigma_{\lambda+1}-\sigma_\lambda)-A_\alpha.$$

But this will always be possible if

$$(\rho_\lambda-\rho_{\lambda+1})\sigma_\lambda>-\rho_{\lambda+1}\sigma_{\lambda+1}+\rho_\lambda\sigma_\lambda+\alpha(\sigma_{\lambda+1}-\sigma_\lambda)-A_\alpha$$

i.e., if

$$(\alpha-\rho_{\lambda+1})(\sigma_\lambda-\sigma_{\lambda+1})>-A_\alpha,$$

a relation which is always true since $\alpha-\rho_{\lambda+1}$ and $\sigma_\lambda-\sigma_{\lambda+1}$ are both positive.

Once more, it holds when $m=\lambda-1$ if

$$[\rho_{\lambda-1}]-[\rho_\lambda]>-\rho_{\lambda+1}\sigma_{\lambda+1}+\rho_\lambda\sigma_\lambda+\alpha(\sigma_{\lambda-1}-\sigma_\lambda)-A_\alpha$$

i.e., if

$$(\rho_\lambda-\rho_{\lambda-1})\tau_{\lambda-1}>-\rho_{\lambda+1}\sigma_{\lambda+1}+\rho_\lambda\sigma_\lambda+\alpha(\sigma_{\lambda-1}-\sigma_\lambda)-A_\alpha$$

i.e., if

$$(\rho_{\lambda-1}-\rho_\lambda)\tau_{\lambda-1}<\rho_{\lambda-1}\sigma_{\lambda-1}-\rho_\lambda\sigma_\lambda-\alpha(\sigma_{\lambda-1}-\sigma_\lambda)+A_\alpha$$

and this can, as in the last case, be shown to be always possible.

Now if the inequality (D) holds, m being greater than λ, it will hold when for m we write $m+1$ provided that

$$[\rho_{m+1}]-[\rho_m]+\rho_{m+1}\sigma_{m+1}-\rho_m\sigma_m>\alpha(\sigma_{m+1}-\sigma_m)$$

i.e., if

$$\rho_m(\sigma_{m+1}-\sigma_m)>\alpha(\sigma_{m+1}-\sigma_m)$$

but

$$\rho_m<\alpha, \quad \sigma_m>\sigma_{m+1}$$

therefore this relation does hold.

But the inequality (D) is true when $m=\lambda+1$. It is therefore true for all larger values of m.*

It can similarly be shown that if the inequality holds, m being less than λ, it will hold when for m we write $m-1$; and that, since it holds when $m=\lambda-1$, it holds for all less values of m.

It is therefore proved universally.

We observe that, as was stated at the outset, the consideration of the case (b) has only introduced a restriction into the conditions of the case (a)—viz.: that the τ's are no longer merely subject to the necessity of lying between consecutive σ's, but must satisfy the closer conditions expressed by the inequalities

$$(\rho_\lambda-\rho_{\lambda+1})\tau_\lambda \begin{cases} >\rho_\lambda\sigma_\lambda-\rho_{\lambda+1}\sigma_{\lambda+1}+\alpha(\sigma_{\lambda+1}-\sigma_\lambda)-A_\alpha \\ <\rho_\lambda\sigma_\lambda-\rho_{\lambda+1}\sigma_{\lambda+1}+\alpha(\sigma_{\lambda+1}-\sigma_\lambda)+A_\alpha \end{cases} \quad . \quad . \quad . \quad . \quad (E)$$

where in the first line α denotes any one of the numbers of the λ^{th}, in the second any one of the $(\lambda+1)^{\text{th}}$ set.

19. We have next to consider the second proposition of page 735, viz.: The condition of (i) is necessary if $\mu-\alpha$ is to have its smallest value.

* It must be observed that this method of proof could not be used to deduce the case $m+1, \lambda+1$ from the case m, λ; for it would not be necessarily true that ρ_m is less than α.

Writing down a series of equations similar to (C) we have

$$\left.\begin{array}{llll}
\overline{\theta(y_1)} & =[\rho_1]+\rho_1\sigma_1=[n-1] & +(n-1)\sigma_1 & +A_{n-1} \\
\overline{\theta(y_2)} & =[\rho_1]+\rho_1\sigma_1=[n-2] & +(n-2)\sigma_1 & +A_{n-2} \\
\&c. & =\&c. \quad =\&c. & & \\
\overline{\theta(y_{k_1})} & =[\rho_1]+\rho_1\sigma_1=[n-k_1] & +(n-k_1)\sigma_1 & +A_{n-k_1}
\end{array}\right\} \text{First set}$$

$$\left.\begin{array}{llll}
\overline{\theta(y_{k_1+1})} & =[\rho_2]+\rho_2\sigma_2=[n-k_1-1]+(n-k_1-1)\sigma_2 & & +A_{n-k_1-1} \\
\&c. & =\&c. \quad =\&c. & &
\end{array}\right\} \text{Second set}$$

$$\&c. \quad =\&c. \quad =\&c. \quad \&c.$$

(F)

and, adding all these lines together,

$$\Sigma\overline{\theta(y)}=[n-1]+[n-2]+\ldots+[0]+(n-1+\ldots+n-k_1)\sigma_1 + (n-k_1-1+\ldots+n-k_2)\sigma_2+\ldots+\Sigma A$$

or

$$\Sigma\overline{\theta y}-\Sigma\overline{q}=(n-1+\ldots+n-k_1)\sigma_1+(n-k_1-1+\ldots+n-k_2)\sigma_2+\ldots+\Sigma A$$

Now, if the condition of (i) were not satisfied, some at least of the signs of equality connecting the first and second vertical columns must have been replaced by the sign $>$; and as those between the second and third column would have remained as before, the equality at the head of this page would have become an inequality—*i.e.*, the value of $\Sigma\overline{\theta y}-\Sigma\overline{q}$ would have been greater than it is—*i.e.*, $\mu-\alpha$ would have been greater.

It only remains to consider the term ΣA.

The smaller we can make this sum, and therefore, all the terms being positive, the smaller we can make each term, the less will be our value of $\mu-\alpha$.

Now from the general equation

$$[\rho_\lambda]+\rho_\lambda\sigma_\lambda=[\alpha]+\alpha\sigma_\lambda+A_\alpha$$

we see that, since $[\rho_\lambda]$ and $[\alpha]$ are integers, A_α consists in general of two parts—an integer and the proper fraction which added to $(\alpha-\rho_\lambda)\sigma_\lambda$ will make it integral.

Now we can make the integral part vanish for every value of α; for to do so will only require a relation between the major term and the other terms of each set; so that, given the degree in x of the major term, those of the others in its set can be written down.

As the conditions (i) only connect with one another the major terms of different sets, this last condition is independent of them and can always be satisfied.

20. To find the value of $\mu-\alpha$ we must investigate the fractional parts.

Considering any set (say the λ^{th}), they are, with the notation of the Lemma (p. 746), of the form

$$\epsilon\frac{(\rho_\lambda-\alpha)m_\lambda}{\mu_\lambda}$$

where α takes each value from $n-k_{\lambda-1}-1$ to $n-k_\lambda$; and $k_\lambda-k_{\lambda-1}=n_\lambda\mu_\lambda$.

But m_λ and μ_λ are prime to each other.

Therefore, by the result of the Lemma, the sum

$$=n_\lambda\frac{\mu_\lambda-1}{2}.$$

So then finally, giving to ΣA its least value, we have

$$\Sigma\overline{\theta y}-\Sigma\overline{q}=\{(n-1)+\ldots+(n-k_1)\}\sigma_1+\{(n-k_1-1)+\ldots+(n-k_2)\}\sigma_2+\ldots+\Sigma\tfrac{1}{2}n(\mu-1)$$

This expression

$$=\tfrac{1}{2}k_1\sigma_1(2n-k_1-1)+\tfrac{1}{2}k_2\sigma_2(2n-k_1-k_2-1)+\ldots+\Sigma\tfrac{1}{2}n(\mu-1).$$

Now $k_1=n_1\mu_1$; $k_2=n_1\mu_1+n_2\mu_2$; &c. = &c.; $n=k_l=n_1\mu_1+n_2\mu_2+\ldots+n_l\mu_l$.

Substituting we obtain

$$\begin{aligned}&n_1m_1\left(\frac{n_1\mu_1-1}{2}+n_2\mu_2+\ldots+n_l\mu_l\right)\\&+n_2m_2\left(\frac{n_2\mu_2-1}{2}+n_3\mu_3+\ldots+n_l\mu_l\right)\\&+\ldots\\&+\Sigma\tfrac{1}{2}n(\mu-1)\\&=\sum_{r>s}n_rm_rn_s\mu_s+\tfrac{1}{2}\Sigma n^2m\mu+\tfrac{1}{2}\Sigma n\mu-\tfrac{1}{2}\Sigma n-\tfrac{1}{2}\Sigma nm.\end{aligned}$$

Now, returning to the values of art. 13 and inserting the numbers A and B for the correction there explained and writing instead of $\Sigma n\mu$ its equivalent n, we have the result following.

The least number of functions in terms of which the sum of any number may be expressed is independent of everything but the form of the function considered (i.e., the form of y given as a function of x by the equation $\chi(y)=0$), and if this equation has

$n_1\mu_1$ roots of the form $y=Cx^{\frac{m_1}{n_1}}+\ldots$, *$n_2\mu_2$ of the form* $y=Cx^{\frac{m_2}{n_2}}+\ldots$, *and so on, the number is*

$$=\sum_{i>r} n_i m_r n_r \mu_r + \tfrac{1}{2}\Sigma n^2 m\mu - \tfrac{1}{2}\Sigma nm - \tfrac{1}{2}\Sigma n - \tfrac{1}{2}n + 1 - A - B \quad . \quad . \quad . \quad . \quad (G)$$

(the last two terms $-A-B$ corresponding to a correction which is in general zero).*

21. It may be well to render these methods and formulæ plainer by applying them to an example. We will choose for this purpose the simple case already considered in the note on p. 734.

Our last formula for the value of $\mu-\alpha$ gives, if we assume that, as in general is the case, the values of A and B are zero, writing

$$m_1=0 \qquad n_1=1 \quad \sigma_1=\tfrac{3}{2} \qquad \mu_1=2$$

$$m_2=-1 \qquad n_2=1 \quad \sigma_2=-1 \qquad \mu_2=1$$

$$\mu-\alpha=3(1)+\tfrac{1}{2}(6-1)-\tfrac{1}{2}(3-1)-\tfrac{1}{2}(2)-\tfrac{3}{2}+1$$
$$=3.$$

We will next find the values of $\overline{q_0}$, $\overline{q_1}$, $\overline{q_2}$, or, as we have written them, [0], [1], [2]. We have

$$\rho_1=2 \text{ or } 1$$
$$\rho_2=0$$

Let us take $\rho_1=2$.†

Then, by the formulæ (F),

$$[2]=[2]$$
$$[1]=[2]+\tfrac{3}{2}-A_1; \text{ so } A_1=\tfrac{1}{2}; \quad [1]=[2]+1$$
$$[0]=[0]$$

* In the most simple case, when

$$y^n+p_1y^{n-1}+\ldots+p_{n-1}y+p_n$$

is the completely general function $(y, x, 1)^n$

$$n_1=n, \quad m_1=1, \quad \mu_1=1$$

and

$$\mu-\alpha=\tfrac{1}{2}n^2-\tfrac{3}{2}n+1=\tfrac{1}{2}(n-1)(n-2)$$
$$=\text{deficiency of general } n\text{-tic curve.}$$

This is a case of the result shown by Professor CAYLEY in the Addition to be universally true.

† We might have taken $\rho_1=1$ with a similar result. This multiplicity of solution will generally occur.

and by the definition of τ

$$[0]-[2]=2\tau_1$$

while from the conditions (E)

$$2\tau_1>3-\tfrac{2}{3}\alpha-A_\infty, \text{ i.e., } >3-5, \text{ or } 3-\tfrac{2}{3}-\tfrac{1}{2}$$
$$<3-\tfrac{2}{3}\alpha+A_\infty, \text{ i.e., } <3$$

so that

$$2\tau_1>0<3$$

whence

$$[0]=[2],\ [2]+1,\ [2]+2,\ [2]+3,$$

and so if the degree of q_2 be denoted by θ that of q_1 is $\theta+1$; and that of q_0 may be either θ, $\theta+1$, $\theta+2$, or $\theta+3$.

We have, then, by art. 13 (i)

$$\alpha=[0]+[1]+[2]+2=3\theta+3,\ 3\theta+4,\ 3\theta+5, \text{ or } 3\theta+6$$

while

$$\mu=n_1\mu_1\{[\rho_1]+\rho_1\sigma_1\}+n_2\mu_2\{[\rho_2]+\rho_2\sigma_2\}$$
$$=2(\theta+3)+\{\theta,\ \theta+1,\ \theta+2,\ \theta+3\}$$
$$=3\theta+6,\ 3\theta+7,\ 3\theta+8,\ 3\theta+9$$

So that, as on the last page,

$$\mu-\alpha=3.$$

We have proved then that the sum of any number of integrals of the form indicated by the fact that they are rationalized by the introduction of y, where

$$y^3+p_2y^2+p_1y+p_0=0,$$

can be reduced to the sum of *three;* the equation of condition being $q_2y^2+q_1y+q_0=0$, where $\overline{q_1}=\overline{q_2}+1$, and $\overline{q_0}$ lies between $\overline{q_2}$ and $\overline{q_2}+3$ inclusive.

Section III.

22. We have shown that the sum of any number whatever of similar functions such as are discussed in this paper can be reduced to an expression algebraical or logarithmic added to a fixed number of such functions whose variables are functions of the variables of the given functions, this fixed number depending only on the form of function considered.

From this a more general theorem may be shown to follow, viz.: that *a similar*

expression may be found for the sum of any number of such functions each multiplied by any rational number positive or negative, integral or fractional.

If all the rational numbers are positive and integral the theorem follows at once by supposing the functions whose sum we have shown how to express to be equal in sets. And this suggests the method of treating the general case when the numbers are any whatever.

Let $\theta \equiv \mu - \alpha =$ fixed number to which the sum of the functions has been shown to be reducible.

Then, by previous work (compare pp. 731, 732).

$$\psi_1(x_1)+\psi_2(x_2)+\ldots+\psi_\alpha(x_\alpha)=v-\{\psi_{\alpha+1}(x_{\alpha+1})+\ldots+(\psi_{\alpha+\theta})(x_{\alpha+\theta})\}$$
$$\psi_1(X_1)+\psi_2(X_2)+\ldots+\psi_{\alpha'}(X_{\alpha'})=V-\{\psi_{\alpha'+1}(X_{\alpha'+1})+\ldots+\psi_{\alpha'+\theta}(X_{\alpha'+\theta})\}$$

where α and α' are any numbers whatever; $x_{\alpha+1}\ldots x_{\alpha+\theta}$ are functions of $x_1\ldots x_\alpha$; and $X_{\alpha'+1}\ldots X_{\alpha'+\theta}$ of $X_1\ldots X_{\alpha'}$, and v, V are algebraical and logarithmic functions.

Subtract: and let the last θ of the terms on the left-hand side of the second be (both as to functional form and variable) identical with those in the bracket in the first. Then, writing β for $\alpha'-\theta$, we have

$$\psi_1(x_1)+\ldots+\psi_\alpha(x_\alpha)-\psi_1(X_1)-\ldots-\psi_\beta(X_\beta)=v-V+\{\psi_{\alpha'+1}X_{\alpha+1}+\ldots+\psi_{\alpha+\theta}(X_{\alpha+\theta})\}.$$

Equate all the functions on the right to zero.

This will give θ relations between the x's and X's.

Now making the functions on the left equal in sets, and dividing by any requisite integer, we have a result which may be written

$$h_1\phi_1(y_1)+h_2\phi_2(y_2)+\ldots+h_m\phi_m(y_m)\equiv W$$

where the ϕ's are similar functions, m is any number whatever, W is an algebraical and logarithmic function of the y's, which are themselves connected by θ relations, and the h's are any numbers whatever.

If we express θ of these variables as functions of the rest and call them z's, putting n for $m-\theta$, we can write

$$h_1\phi_1(y_1)+h_2\phi_2(y_2)+\ldots+h_n\phi_n(y_n)=w+k_1\phi'_1(z_1)+\ldots+k_\theta\phi'_\theta(z_\theta).$$

Or making, as we may, the k's each = unity we have shown how to find the expression required.*

* The subscript letters attached here, and not before, to the functional symbols introduce no novelty. They are only intended to suggest the fact that what we have written $\psi(x_1)$, $\psi(x_2)\ldots$ are really $\psi(x_1, y_1)$, $\psi(x_2, y_2)$, $\ldots$; while y_1 and $y_2\ldots$ are not necessarily the same functions of x_1, $x_2\ldots$ This has not been hitherto overlooked, it is only more clearly put in evidence now.

23. We may conveniently investigate at this point, as a corollary to previous work, the conditions necessary that the 'algebraic and logarithmic function' often referred to already should become a constant; in other words, that the term involving Θ in the expression of ABEL's theorem should disappear, and with it the arbitrary quantities $a_1, a_2 \ldots$

We will assume $F_0(x)=1$ for the sake of simplicity, and have therefore the formula of Art. 9.

The first condition is that

$$f_2(x)=1 \quad \ldots \quad (A)$$

for otherwise the terms contributed by it to Θ will introduce the arbitrary quantities a.

Next, we must have

$$C_2 \Sigma \frac{f_1(x,y)}{\chi'(y)} \log \theta y = 0$$

or, which comes to the same effect,

$$C_2 \Sigma \frac{f_1(xy)}{\chi'(y)} \cdot \frac{\delta \theta y}{\theta y} = 0$$

and since $\overline{\delta \theta y} = \overline{\theta y}$, δ indicating differentiation with respect to a's, and consequently not altering the degree of a function in x,

$$\overline{\left(\frac{\delta \theta y}{\theta y}\right)} = 0$$

and the condition to be satisfied is

$$\overline{\left(\frac{f_1(xy)}{\chi'(y)}\right)} < -1$$

when, for y, any whatever of the series $y_1, y_2 \ldots y_n$ has been substituted.

Now $f_1(x\,y)$, being integral and rational, can be expanded in the form

$$\sum_{r=0}^{r=n-1} P_r y^r.$$

We require then that, for all values of k and r from 0, to $n-1$

$$\overline{P_r} + r\overline{y_k} - \overline{\chi'(y_k)} < -1.$$

Now

$$\chi'(y_k) = (y_k - y_1)(y_k - y_2) \ldots (y_k - y_{k-1})(y_k - y_{k+1}) \ldots (y_k - y_n)$$

whence

$$\overline{\chi'(y_k)} = \overline{y_1} + \overline{y_2} + \ldots + \overline{y_{k-1}} + (n-k)\overline{y_k}$$

so that

$$\overline{P_r} < -1 + \overline{y_1} + \ldots + \overline{y_{k-1}} + (n-k-r)\overline{y_k}$$

Now, to write $k+1$ for k is to change the right hand side of this inequality by $\overline{y_k}-(n-k-r)\overline{y_k}+(n-k-r-1)\overline{y_{k+1}}$; *i.e.*, by $(n-k-r-1)(\overline{y_{k+1}}-\overline{y_k})$.

This is negative if $k<n-r-1$

vanishes if $k=n-r-1$

is positive if $k>n-r-1$.

So there is a minimum value when $k=n-r-1$, and we must therefore have

$$P_r<-1+\overline{y_1}+\overline{y_2}+\ldots+\overline{y_{n-r-1}}.$$

Let $n-r-1=k_\alpha+\beta$ (and lie between k_α and $k_{\alpha+1}$),

then
$$\overline{P_r}<-1+n_1m_1+\ldots+n_\alpha m_\alpha+\beta\frac{m_{\alpha+1}}{\mu_{\alpha+1}}$$

Therefore

$$\overline{P_r}\leqq E\left\{\sum_{i=1}^{i=\alpha}n_im_i+\beta\frac{m_{\alpha+1}}{\mu_{\alpha+1}}\right\}-1. \quad \ldots\ldots\ldots \quad (B)$$

If this is to be true whatever r is, it must hold when we put $\alpha=0$;

wherefore
$$\overline{P_r}\leqq E\left(\beta\frac{m_1}{\mu_1}\right)-1\leqq E(\beta\sigma_1)-1$$

where r is one of the numbers $n-1$, $n-2$, $\ldots$ $n-k_1$ and β is less than k_1: for $\beta=n-r-1$.

Now $\overline{P_r}$ cannot be negative, therefore the smallest value assignable to β is the least which makes

$$E(\beta\sigma_1)=1;\ \textit{i.e.},\ \text{is } (\beta'\equiv)E\left(\frac{1}{\sigma}\right)+1.$$

We must then have $P_{n-\beta'-1}\,y^{n-\beta'-1}$ *as the highest term in* $f_1(x, y)$.

This condition, necessary—and, as we see without difficulty, sufficient also; for the values assigned by equation (B) to $\overline{P_r}$ are clearly positive when α is greater than zero—can always be satisfied unless $\beta'=n$.

This can only happen in two cases, viz.: when $\sigma=\frac{1}{n-1}$ or $\sigma=\frac{1}{n}$. In these two cases it can be easily shown that a *single* integral of the given form can be expressed by

means of algebraic and logarithmic functions; so that ABEL'S theorem becomes unnecessary.

Except then in these two cases it is always possible by satisfying the conditions (A) and (B) to render the sum of the series of functions equal to a constant.*

The number of arbitrary constants, being equal to the number of relations connecting the variables of the functions which we sum, will by art. 20 (G) be

$$\sum_{r>r} n,m,n,\mu,+\tfrac{1}{2}\Sigma n^2 m\mu-\tfrac{1}{2}\Sigma nm-\tfrac{1}{2}\Sigma n-\tfrac{1}{2}n+1.$$

It is not necessary that we should assume $F_0(x)=1$ for the correctness of the processes of the last two pages.

Our equations will be the same if for any other reason $F_0(x)$ disappears from the general formula, and reduces it to the case of art. 9.

But this will happen if in the denominator of $\frac{1}{f_3(x)}\Sigma\frac{f_1(x,y)}{\chi'(y)}\log\theta y$ there is no factor also occurring in $F_0(x)$; and this will be so *if* $F_0(x)$ *and* $\frac{\chi'(y)}{f_1(x,y)}$ *do not vanish for any the same value of* x.

If this condition hold the results just arrived at will remain true.

APPENDIX.

LEMMA.

To find the values (i) *of the integral parts,* (ii) *of the fractional parts,* (iii) *of the complements to the fractional parts of the series of terms*

$$\frac{a}{n},\ \frac{a+b}{n},\ \frac{a+2b}{n},\ \ldots\ \frac{a+(n-1)b}{n}$$

where n *is a positive integer, and* a *and* b *are integers positive or negative.*

By the *integral part* of a term we mean the integer next less than or equal to it; by the *fractional part* that positive fraction (zero included) which added to the *integral part* gives the number; by the *complement of the fractional part* that fraction which added to the given number produces the next higher integer.

Let these functions of the term be denoted by the symbols $E\ \epsilon\ \epsilon'$.

* A notable particular case is that in which $f_1(x, y)$ consists of a single term, $x^k y^m$; where m is so chosen as to satisfy the condition (B) above, and k so as to satisfy the equation (i) of the last page.

Then, by the theory of numbers, if b and n are prime, the integers

$$n\epsilon\frac{a}{n},\ n\epsilon\frac{a+b}{n},\ \ldots\ n\epsilon\frac{a+\overline{n-1}.\,b}{n}$$

will be (in some order) the series 0, 1, 2 . . . , $n-1$; while if b and n are divisible by c, c being their greatest common measure, the integers

$$n\epsilon\frac{a}{n},\ n\epsilon\frac{a+b}{n},\ \ldots,\ n\epsilon\frac{a+\overline{n-1}.\,b}{n}$$

form an arithmetical progression whose common difference is c, repeated c times; and the smallest term of this progression is the remainder when c is divided into a.

If this remainder be called d we have

$$\sum_{l=0}^{l=n-1}\epsilon\frac{a+lb}{n}=\frac{c}{n}\left\{d+(d+c)+(d+2c)+\ \ldots\ \text{to}\ \frac{n}{c}\ \text{terms}\right\}$$

$$=d+\frac{n-c}{2}$$

whence

$$\sum_{l=0}^{l=n-1}\epsilon'\frac{a+lb}{n}=n-\sum_{l=0}^{l=n-1}\epsilon\frac{a+lb}{n}$$

$$=-d+\frac{n+c}{2}$$

and

$$\sum_{l=0}^{l=n-1}\mathrm{E}\frac{a+lb}{n}=\Sigma\frac{a+lb}{n}+d-\frac{n+c}{2}$$

Corollary i.

If c the greatest common factor of b and n also divides a, then $d=0$, and we have the simpler forms

$$\Sigma\epsilon'=\frac{n+c}{2},\ \Sigma\epsilon=\frac{n-c}{2}.$$

Corollary ii.

The sum of the fractional parts of any n *terms of the series* (*repetitions being allowed*) *differs from the sum of the fractional parts of the values of the same terms when* a *is put equal to zero, by an integer.*

For, if the sum of the coefficients of b in the numerators of the n terms be λ, then

$$\Sigma\epsilon_1 = a + \frac{\lambda b}{n} - \Sigma E_1 \text{ in the first case}$$

$$\Sigma\epsilon_2 = \frac{\lambda b}{n} - \Sigma E_2 \text{ in the second}$$

(the notation being obvious)

wherefore $\Sigma\epsilon - \Sigma\epsilon' = a + \Sigma E_2 - \Sigma E_1 =$ an integer

which is the required result.

List of Errata.

In Abel's Mem. the following slighter mistakes should be corrected :—

Page 184, ll. 12, 13,	for F	read T
192, l. 4,	for $\theta_1 x - \beta$	read $(x-\beta)^r$.
200, l. 3,	for hy^m	read $hy^{m'}$.
207, l. 9,	for χy	read $\chi' y$.
231, l. 2,	for z_1	read z_2.
231, l. 3,	for z_2	read z_3.
233,	for γ	read r throughout.
240,	for s^m	read s_m throughout.
243, l. 2,	for $n\delta_{2,r}$	read $n\delta_{2,s}$.
252, last line,	for s_{n-1}	read s_{n-1}.
255, last line but one,	for z	read 2.

There are besides these the inaccuracies referred to by M. Libri (the editor of the paper) as occurring on pp. 226–8.

These are too numerous to be treated otherwise than by re-writing the pages, which has therefore been done ; and they immediately follow.

" Alors l'equation (92) donnera les suivantes :—

$$f(12) = f(11) - \tfrac{4}{3} - A_0^{i}, \quad \text{donc } A_0^{i} = \tfrac{3}{3} \quad f(12) = f(11) - 2.$$
$$f(10) = f(11) + \tfrac{4}{3} - A_2^{i}, \quad \text{donc } A_2^{i} = \tfrac{1}{3} \quad f(10) = f(11) + 1.$$
$$f(9) = f(6) - \tfrac{3}{5} - A_3^{ii}, \quad \text{donc } A_3^{ii} = \tfrac{3}{5} \quad f(9) = f(6) - 1.$$
$$f(8) = f(6) - \tfrac{2}{5} - A_4^{ii}, \quad \text{donc } A_4^{ii} = \tfrac{3}{5} \quad f(8) = f(6) - 1.$$
$$f(7) = f(6) - \tfrac{1}{5} - A_5^{ii}, \quad \text{donc } A_5^{ii} = \tfrac{4}{5} \quad f(7) = f(6) - 1.$$
$$f(5) = f(6) + \tfrac{1}{5} - A_7^{ii}, \quad \text{donc } A_7^{ii} = \tfrac{1}{5} \quad f(5) = f(6).$$
$$f(3) = f(4) - \tfrac{1}{2} - A_9^{iii}, \quad \text{donc } A_9^{iii} = \tfrac{1}{2} \quad f(3) = f(4) - 1.$$
$$f(2) = f(4) - 1 - A_{10}^{iii}, \quad \text{donc } A_{10}^{iii} = 0 \quad f(2) = f(4) - 1.$$
$$f(1) = f(4) - \tfrac{3}{2} - A_{11}^{iii}, \quad \text{donc } A_{11}^{iii} = \tfrac{1}{2} \quad f(1) = f(4) - 2.$$

"Pour trouver maintenant $f(0)$, $f(4)$, $f(6)$, $f(11)$, il faut chercher les limites de θ_1, θ_2, θ_3, θ_4.

"Or les équations (103), qui déterminent ces limites donnent

$$\theta_1 > \frac{11-\alpha_1}{5} - \frac{3A_\beta^{\,i}}{17}\ ;\ \text{d'où}\ \theta_1 > -\frac{1}{5} - \frac{2}{17}\ ;\ 0\ ;\ \frac{1}{5} - \frac{1}{17}.$$

$$\theta_1 < \frac{11-\alpha_1}{5} + \frac{3A_\beta^{\,ii}}{17}\ ;\ \text{d'où}\ \theta_1 < \frac{2}{5} + \frac{6}{5.17}\ ;\ \frac{3}{5} + \frac{9}{5.17}\ ;\ \frac{4}{5} + \frac{12}{5.17}\ ;\ 1\ ;\ \frac{6}{5} + \frac{3}{5.17}.$$

"Il suit de là que

$$\theta_1 > \frac{12}{55} < \frac{40}{55}$$

"On a aussi

$$\theta_2 > \frac{6-\alpha_2}{2} - \frac{5A_\beta^{\,ii}}{7}\ ;\ \text{d'où}\ \theta_2 > \ldots,\ \frac{1}{2} - \frac{1}{7}$$

$$\theta_2 < \frac{6-\alpha_2}{2} + \frac{5A_\beta^{\,iii}}{2}\ ;\ \text{d'où}\ \theta_2 < \frac{2}{2},\ \frac{3}{2} + \frac{5}{14},\ \ldots$$

"Il suit que

$$\theta_2 > \frac{5}{14} < 1$$

$$\theta_3 > \frac{4-\alpha_3}{4} - \frac{A_\beta^{\,iii}}{2}\ ;\ \text{d'où}\ \theta_3 > 0,\ \frac{1}{4} - \frac{1}{4},\ \frac{2}{4},\ \frac{3}{4} - \frac{1}{4}$$

$$\theta_3 < \frac{4-\alpha_3}{4} + \frac{A_\beta^{\,iv}}{2}\ ;\ \text{d'où}\ \theta_3 < 1$$

"Il suit que

$$\theta_3 > \tfrac{1}{2} < 1.$$

"Maintenant l'équation (97) donne

$$f(\rho_m) - f(\rho_{m-1}) > (\rho_{m-1} - \rho_m)(\theta''_{m-1}\sigma_{m-1} + \overline{1-\theta''_{m-1}}\sigma_m)$$
$$f(\rho_m) - f(\rho_{m-1}) < (\rho_{m-1} - \rho_m)(\theta'_{m-1}\sigma_{m-1} + \overline{1-\theta'_{m-1}}\sigma_m)$$

où θ''_{m-1} est la plus petite, et θ'_{m-1} la plus grande valeur de θ_{m-1}; donc on trouvera, en faisant $m=2, 3, 4$,

$$f(6) - f(11) > 5(\tfrac{2}{5}\cdot\tfrac{4}{5} + \overline{1-\tfrac{2}{5}\cdot\tfrac{4}{5}}\cdot\tfrac{1}{5}\ );\ (=1+\tfrac{8}{25})$$
$$f(6) - f(11) < 5(\tfrac{4}{5}\cdot\tfrac{4}{5} + \overline{1-\tfrac{4}{5}\cdot\tfrac{4}{5}}\cdot\tfrac{1}{5}\ );\ (=3+\tfrac{24}{25})$$
$$f(4) - f(6)\ > 2(\tfrac{2}{4}\cdot\tfrac{1}{2} - \overline{1-\tfrac{2}{4}\cdot\tfrac{1}{2}}\ );\ (=-\tfrac{1}{2})$$
$$f(4) - f(6)\ < 2(1\cdot\tfrac{1}{2} - \overline{1-1}\cdot\tfrac{1}{2}\ );\ (=\tfrac{2}{2})$$
$$f(0) - f(4)\ > 4(\tfrac{1}{2}\cdot\overline{-\tfrac{1}{2}} + \overline{1-\tfrac{1}{2}}\cdot\overline{-1});\ (=-3)$$
$$f(0) - f(4)\ < 4(1\cdot\overline{-\tfrac{1}{2}} + \overline{1-1}\cdot\overline{-1});\ (=-2)$$

donc on aura pour $f(6)-f(11), f(4)-f(6), f(0)-f(4)$ les valeurs suivantes :—

$$f(6)-f(11)=2, 3. \quad f(4)-f(6)=0. \quad f(0)-f(4)=-3, -2.$$

d'où

$$f(11)=f(6)-2, f(6)-3\,; f(4)=f(6);\ f(0)=f(6)-3, f(6)-2$$
$$f(12)=f(6)-4, f(6)-5\,; f(10)=f(6)-1, f(6)-2$$
$$f(9)=f(6)-1\,; f(8)=f(6)-1\,; f(7)=f(6)-1, f(5)=f(6)$$
$$f(3)=f(6)-1\,; f(2)=f(6)-1\,; f(1)=f(6)-2$$

"En exprimant donc toutes ces quantités par $f(12)$ on voit que les fonctions $q_{12}, q_{11}, \ldots q_0$ sont respectivement des degrés suivants :—

(12) (11) (10) (9) (8) (7) (6) (5) (4) (3) (2) (1) (0)
θ, $\theta+2$, $\theta+3$, $\theta+3$, $\theta+3$, $\theta+3$, $\theta+4$, $\theta+4$, $\theta+4$, $\theta+3$, $\theta+3$, $\theta+2$, $(\theta+2, \theta+1)$

ou

(12) (11) (10) (9) (8) (7) (6) (5) (4) (3) (2) (1) (0)
θ, $\theta+2$, $\theta+3$, $\theta+4$, $\theta+4$, $\theta+4$, $\theta+5$, $\theta+5$, $\theta+5$, $\theta+4$, $\theta+4$, $\theta+3$, $(\theta+3, \theta+2)$

où θ est le degré de la fonction q_{12}.

"De là suit que

$$\alpha=f(0)+f(1)+\ldots+f(12)+12=13\theta+47,\ 13\theta+48$$
$$13\theta+57,\ 13\theta+58$$

et

$$\mu=n'\mu'\left(f(\rho_1)+\rho_1\frac{m'}{\mu'}\right)+\ldots+\ldots+\ldots$$
$$=3f(11)+44+5f(6)+6+4f(4)-8+f(0)$$

c'est à dire

$$\mu=13\theta+95,\ \ 13\theta+96$$
$$13\theta+105,\ 13\theta+106$$

"La valeur de $\mu-\alpha$ deviendra donc

$$\mu-\alpha=38$$

comme nous avons trouvé plus haut."

Addition to Mr. Rowe's *Memoir.*

By Professor Cayley, *F.R.S.*

Received May 27,—Read June 10, 1880.

In Abel's general theorem y is an irrational function of x determined by an equation $\chi(y)=0$ (or say $\chi(x, y)=0$) of the order n as regards y: x and it was shown by him that the sum of any number of the integrals considered may be reduced to a sum of γ integrals; where γ is a determinate number depending only on the form of the equation $\chi(x, y)=0$, and given in his equation (62) p. 206: viz., if (solving the equation so as to obtain from it developments of y in descending series of powers of x) we have*

$$\begin{aligned} &n_1\mu_1 \text{ series each of the form } y=x^{\frac{m_1}{\mu_1}}+\ldots, \\ &n_2\mu_2 \quad ,, \qquad\qquad ,, \qquad y=x^{\frac{m_2}{\mu_2}}+\ldots, \\ &\qquad\vdots \qquad\qquad\qquad\quad \vdots \\ &n_k\mu_k \quad ,, \qquad\qquad ,, \qquad y=x^{\frac{m_k}{\mu_k}}+\ldots, \end{aligned}$$

(so that $n=n_1\mu_1+n_2\mu_2\ldots+n_k\mu_k$), then γ is a determinate function of n_1, m_1, μ_1; n_2, m_2, μ_2; $\ldots$ n_k, m_k, μ_k.

Mr. Rowe has expressed Abel's γ in the following form, viz., assuming

$$\frac{m_1}{\mu_1}>\frac{m_2}{\mu_2}\ldots>\frac{m_k}{\mu_k},$$

* The several powers of x have coefficients: the form really is $y=A_1x^{\frac{m_1}{\mu_1}}+\ldots$, which is regarded as representing the μ_1 different values of y obtained by giving to the radical $x^{\frac{1}{\mu_1}}$ each of its μ_1 values, and the corresponding values to the radicals which enter into the coefficients of the series: and (so understanding it) the meaning is that there are n_1 such series each representing μ_1 values of y. It is assumed that the series contains *only* the radical $x^{\frac{1}{\mu_1}}$, that is, the indices after the leading index $\frac{m_1}{\mu_1}$ are $\frac{m_1-1}{\mu_1}$, $\frac{m_1-2}{\mu_1}$, $\ldots$; a series such as $y=A_1x^{\frac{1}{2}}+B_1x^{\frac{1}{3}}+\ldots$, depending on the two radicals $x^{\frac{1}{2}}$, $x^{\frac{1}{3}}$ represents 15 different values, and would be written $y=A_1x^{\frac{m_1}{\mu_1}}+\ldots$, or the values of m_1 and μ_1 would be 20 and 15 respectively: in a case like this where $\frac{m_1}{\mu_1}$ is not in its least terms, the number of values of the leading coefficient A_1 is equal, not to μ_1, but to a submultiple of μ_1. But the case is excluded by Abel's assumption that $\frac{m_1}{\mu_1}, \frac{m_2}{\mu_2}\ldots$, are fractions each of them in its least terms.

then this expression is

$$\gamma=\sum_{s>r} n_r m_r n_s \mu_s+\tfrac{1}{2}\Sigma n^2 m\mu-\tfrac{1}{2}\Sigma nm-\tfrac{1}{2}\Sigma n-\tfrac{1}{2}n+1,$$

or what is the same thing, for n writing its value $\Sigma n\mu$,

$$\gamma=\sum_{s>r} n_r m_r n_s \mu_s+\tfrac{1}{2}\Sigma n^2 m\mu-\tfrac{1}{2}\Sigma nm-\tfrac{1}{2}\Sigma n\mu-\tfrac{1}{2}\Sigma n+1,$$

where in the first sum r, s have each of them the values 1, 2, ... k, subject to the condition $s>r$; in each of the other sums n, m, and μ are considered as having the suffix r, which has the values 1, 2, ... k.

It is a leading result in RIEMANN'S theory of the Abelian integrals that γ is the deficiency (Geschlecht) of the curve represented by the equation $\chi(x, y)=0$: and it must consequently be demonstrable *à posteriori* that the foregoing expression for γ is in fact = deficiency of curve $\chi(x, y)=0$. I propose to verify this by means of the formulæ given in my paper "On the Higher Singularities of a Plane Curve," Quart. Math. Jour., vol. vii., pp. (1866) 212–222.

It is necessary to distinguish between the values of $\frac{m}{\mu}$ which are $>$, $=$, and <1; and to fix the ideas I assume $k=7$, and

$$\frac{m_1}{\mu_1}, \frac{m_2}{\mu_2}, \frac{m_3}{\mu_3} \text{ each } >1,$$

$$\frac{m_4}{\mu_4}=1; \text{ say } m_4=\mu_4=\lambda; \text{ and } n_4=\theta,$$

$$\frac{m_5}{\mu_5}, \frac{m_6}{\mu_6}, \frac{m_7}{\mu_7} \text{ each } <1,$$

but it will be easily seen that the reasoning is quite general. I use Σ' to denote a sum in regard to the first set of suffixes 1, 2, 3, and Σ'' to denote a sum in regard to the second set of suffixes 5, 6, 7. The foregoing value of n is thus

$$n=\Sigma' n\mu+\lambda\theta+\Sigma'' n\mu.$$

Introducing a third coordinate z for homogeneity, the equation $\chi(x, y)=0$ of the curve will be

$$0=\left(yz^{\frac{m_1}{\mu_1}-1}-x^{\frac{m_1}{\mu_1}}\right)^{n_1\mu_1}\ldots\left(y-x^{\frac{\lambda}{\lambda}}\right)^{\lambda\theta}\left(y-x^{\frac{m_7}{\mu_7}}z^{1-\frac{m_7}{\mu_7}}\right)\ldots$$

where it is to be observed that $(\quad)^{n_1\mu_1}$ is written to denote the product of $n_1\mu_1$ different series each of the form $yz^{\frac{m_1}{\mu_1}-1}-A_1x^{\frac{m_1}{\mu_1}}\ldots$; these divide themselves into n_1 groups, each a product of μ_1 series; and in each such product the μ_1 coefficients A_1 are in general the

μ_1 values of a function containing a radical $a^{\frac{1}{\mu_1}}$, and are thus different from each other: it is in what follows in effect assumed not only that this is so, but that all the $n_1\mu_1$ coefficients A_1 are different from each other:* the like remarks apply to the other factors. It applies in particular to the term $\left(y-x^{\frac{\lambda}{\lambda}}\right)^{\lambda\theta}$, viz., it is assumed that the coefficients A in the $\lambda\theta$ series $y=Ax^{\frac{\lambda}{\lambda}}+\ldots$, are all of them different from each other. These assumptions as to the leading coefficients really imply ABEL's assumption that $\frac{m_1}{\mu_1},\ldots\frac{m_2}{\mu_2}$ are all of them fractions in their least terms, and in particular that $\frac{\lambda}{\lambda}$ is a fraction in its least terms, viz., that $\lambda=1$: I retain however for convenience the general value λ, putting it ultimately $=1$.

In the product of the several infinite series the terms containing negative powers all disappear of themselves, and the product is a rational and integral function $F(x, y, z)$ of the coordinates, which on putting therein $z=1$ becomes $=\chi(x, y)$. The equation of the curve thus is $F(x, y, z)=0$; and the order is $=\frac{m_1}{\mu_1}n_1\mu_1+\ldots+\lambda\theta+n_2\mu_2+\ldots$, $=m_1n_1+\ldots+\lambda\theta+n_2\mu_2+\ldots$; viz., if K is the order of the curve $\chi(x, y)=0$, then $K=\Sigma'nm+\lambda\theta+\Sigma''n\mu$.

The curve has singularities (singular points) at infinity, that is, on the line $z=0$: viz.—

First, a singularity at $(z=0, x=0)$, where the tangent is $x=0$, and which (writing for convenience $y=1$) is denoted by the function

$$\left(z-x^{\frac{m_1}{m_1-\mu_1}}\right)^{n_1(m_1-\mu_1)}\ldots;$$

where observe that the expressed factor indicates n_1 branches $\left(z-x^{\frac{m_1}{m_1-\mu_1}}\right)^{m_1-\mu_1}$, or say $n_1(m_1-\mu_1)$ partial branches $z-x^{\frac{m_1}{m_1-\mu_1}}$, that is $n_1(m_1-\mu_1)$ partial branches $z=A_1x^{\frac{m_1}{m_1-\mu_1}}+\ldots$, with in all $n_1(m_1-\mu_1)$ distinct values of A_1; and the like as regards the unexpressed factors with the suffixes 2 and 3.

Secondly, a singularity at $(z=0, y=0)$, where the tangent is $y=0$, and which (writing for convenience $x=1$) is denoted by the function

$$\left(z-y^{\frac{\mu_2}{\mu_2-m_2}}\right)^{n_2(\mu_2-m_2)}\ldots;$$

where observe that the expressed factor indicates n_2 branches $\left(z-y^{\frac{\mu_2}{\mu_2-m_2}}\right)^{\mu_2-m_2}$, or say

* This assumption is virtually made by ABEL, p. 198, in the expression "alors on aura en général, excepté quelques cas particuliers que je me dispense de considérer: $h(y'-y'')=hy'$, &c.": viz., the meaning is that the degree of y' being greater than or equal to that of y'', then the degree of $y'-y''$ is equal to that of y'': of course when the degrees are equal, this implies that the coefficients of the two leading terms must be unequal.

$n_5(\mu_5-m_5)$ partial branches $z-y^{\frac{\mu_5}{\mu_5-m_5}}$, that is $n_5(\mu_5-m_5)$ partial branches $z=A_5y^{\frac{\mu_5}{\mu_5-m_5}}+\ldots$, with in all $n_5(\mu_5-m_5)$ distinct values of A_5: and the like as regards the unexpressed factors with the suffixes 6 and 7.

Thirdly, singularities at the θ points ($z=0$, $y-Ax=0$), A having here θ distinct values, at any one of which the tangent is $y-Ax=0$, and which are denoted by the function

$$\left(y-x^{\frac{\lambda}{\lambda}}\right)^{\lambda\theta}:$$

but in the case ultimately considered λ is $=1$; and these are then the θ ordinary points at infinity, ($z=0$, $y-Ax=0$).

According to the theory explained in my paper above referred to, these several singularities are together equivalent to a certain number $\delta'+\kappa'$ of nodes and cusps, viz., we have

$$\begin{aligned}\delta' &= \tfrac{1}{2}M-\tfrac{3}{2}\Sigma(\alpha-1)\\ \kappa' &= \qquad\Sigma(\alpha-1),\end{aligned}$$

hence

$$\delta'+\kappa'=\tfrac{1}{2}M-\tfrac{1}{2}\Sigma(\alpha-1)$$

and (assuming that there are no other singularities) the deficiency

$$\tfrac{1}{2}(K-1)(K-2)-\delta'-\kappa'$$

is

$$=\tfrac{1}{2}(K-1)(K-2)-\tfrac{1}{2}M+\tfrac{1}{2}\Sigma(\alpha-1)$$

this should be equal to the before-mentioned value of γ, viz., we ought to have

$$(K-1)(K-2)-M+\Sigma(\alpha-1)=2\sum_{r>s}n_rm_sn_r\mu_s+\Sigma n^2m\mu-\Sigma nm-\Sigma n\mu-\Sigma n+2$$

or, as it will be convenient to write it,

$$M=K^2-3K+\Sigma(\alpha-1)-2\sum_{r>s}n_rm_sn_r\mu_s-\Sigma n^2m\mu+\Sigma nm+\Sigma n\mu+\Sigma n,$$

which is the equation which ought to be satisfied by the values of M and $\Sigma(\alpha-1)$ calculated, according to the method of my paper, for the foregoing singularities of the curve.

We have as before

$$K=\Sigma' nm+\Sigma'' n\mu+\theta\lambda.$$

The term $\sum_{r>s}n_rm_sn_r\mu_s$, written at length, is

$$\begin{aligned}
= \quad & n_1m_1(n_2\mu_2+n_3\mu_3+\theta\lambda+n_5\mu_5+n_6\mu_6+n_7\mu_7)\\
+ & n_2m_2(\quad n_3\mu_3+\theta\lambda+n_5\mu_5+n_6\mu_6+n_7\mu_7)\\
+ & n_3m_3(\quad \theta\lambda+n_5\mu_5+n_6\mu_6+n_7\mu_7)\\
+ & \ \theta\lambda\ (\quad n_5\mu_5+n_6\mu_6+n_7\mu_7)\\
+ & n_5m_5(\quad n_6\mu_6+n_7\mu_7)\\
+ & n_6m_6(\quad n_7\mu_7),
\end{aligned}$$

which is

$$=\Sigma' n_r m_r n_s \mu_s + \theta\lambda(\Sigma' nm + \Sigma'' n\mu) + \Sigma' nm . \Sigma'' n\mu + \sum_{s>r}{}'' n_r m_r n_s \mu_s.$$

We have moreover

$$\begin{aligned}
\Sigma n^2 m\mu &= \Sigma' n^2 m\mu + \theta^2\lambda^2 + \Sigma'' n^2 m\mu,\\
\Sigma nm &= \Sigma' nm + \theta\lambda + \Sigma'' nm,\\
\Sigma n\mu &= \Sigma' n\mu + \theta\lambda + \Sigma'' n\mu,\\
\Sigma n &= \Sigma' n + \theta + \Sigma'' n.
\end{aligned}$$

We next calculate $\Sigma(\alpha-1)$.

For the singularity

$$\left(z-x^{\frac{m_1}{m_1-\mu_1}}\right)^{n_1(m_1-\mu_1)} \ldots$$

each branch $\left(z-x^{\frac{m_1}{m_1-\mu_1}}\right)^{m_1-\mu_1}$ gives $\alpha=m_1-\mu_1$, and the value of $\Sigma(\alpha-1)$ for this singularity is $n_1(m_1-\mu_1-1)+n_2(m_2-\mu_2-1)+n_3(m_3-\mu_3-1)$, which is

$$=\Sigma' nm - \Sigma' n\mu - \Sigma' n.$$

For the singularity

$$\left(z-y^{\frac{\mu_5}{\mu_5-m_5}}\right)^{n_5(\mu_5-m_5)} \ldots$$

each branch $\left(z-y^{\frac{\mu_5}{\mu_5-m_5}}\right)^{\mu_5-m_5}$ gives $\alpha=\mu_5-m_5$, and the value of $\Sigma(\alpha-1)$ for this singularity is $n_5(\mu_5-m_5-1)+n_6(\mu_6-m_6-1)+n_7(\mu_7-m_7-1)$, which is

$$=\Sigma'' n\mu - \Sigma'' nm - \Sigma'' n.$$

For each of the θ singularities

$$\left(y-x^{\frac{\lambda}{\lambda}}\right)^{\lambda\theta}$$

we have $\alpha=\lambda$ and the value of $\Sigma(\alpha-1)$ is $=\theta(\lambda-1)$: this is $=0$ for the value $\lambda=1$, which is ultimately attributed to λ.

The complete value of $\Sigma(\alpha-1)$ is thus

$$=\Sigma' nm-\Sigma'' nm-\Sigma' n\mu+\Sigma'' n\mu-\Sigma' n-\Sigma'' n-\theta\lambda-\theta.$$

Substituting all these values we have

$$\begin{aligned}
\mathrm{M}=&(\Sigma' nm+\Sigma'' n\mu)^2+2\theta\lambda(\Sigma' nm+\Sigma'' n\mu)+(\theta\lambda)^2\\
&-3(\Sigma' nm+\Sigma'' n\mu)-3\theta\lambda\\
&+\Sigma' nm-\Sigma'' nm-\Sigma' n\mu+\Sigma'' n\mu-\Sigma' n-\Sigma'' n+\theta\lambda-\theta\\
&-2\sum_{s>r}n_r m_r n_s\mu_s-2\theta\lambda(\Sigma' nm+\Sigma'' n\mu)-2\Sigma' nm.\Sigma'' n\mu-2\sum_{s>r}n_r m_r n_s\mu_s\\
&-\Sigma' n^2 m\mu-(\theta\lambda)^2-\Sigma'' n^2 m\mu\\
&+\Sigma' nm+\theta\lambda+\Sigma'' nm\\
&+\Sigma' n\mu+\theta\lambda+\Sigma'' n\mu\\
&+\Sigma' n+\theta+\Sigma'' n,
\end{aligned}$$

or reducing

$$\begin{aligned}
\mathrm{M}=&(\Sigma' nm)^2-\Sigma' nm-\Sigma' n^2 m\mu-2\sum_{s>r}n_r m_r n_s\mu_s,\\
&+(\Sigma'' n\mu)^2-\Sigma'' n\mu-\Sigma'' n^2 m\mu-2\sum_{s>r}n_r m_r n_s\mu_s;
\end{aligned}$$

and it is to be shown that the two lines of this expression are in fact the values of M belonging to the singularities $\left(z-x^{\frac{m_1}{m_1-\mu_1}}\right)^{n_1(m_1-\mu_1)}\ldots$, and $\left(z-y^{\frac{\mu_1}{\mu_1-m_1}}\right)^{n_1(\mu_1-m_1)}\ldots$ respectively. We assume $\lambda=1$, and there is thus no singularity $\left(y-x^{\frac{\lambda}{\lambda}}\right)^{\lambda\theta}$.

I recall that, considering the several partial branches which meet at a singular point, M denotes the sum of the number of the intersections of each partial branch by every other partial branch (so that for each pair of partial branches the intersections are to be counted *twice*). Supposing that the tangent is $x=0$, and that for any two branches we have $z_1=A_1x^{p_1}$, $z_2=A_2x^{p_2}$ (where p_1, p_2 are each equal to or greater than 1), then if $p_2=p_1$, and $z_1-z_2=(A_1-A_2)x^{p_1}$ where A_1-A_2 not$=0$ (an assumption which has been already made as regards the cases about to be considered), then the number of intersections is taken to be $=p_1$; and if p_1 and p_2 are unequal, then *taking p_2 to be the greater of them*, the leading term of z_1-z_2 is $=A_1x^{p_1}$, and the number of intersections is taken to be $=p_1$; viz., in the case of unequal exponents, it is equal to the smaller exponent.

Consider now the singularity $\left(z-x^{\frac{m_1}{m_1-\mu_1}}\right)^{n_1(m_1-\mu_1)}\ldots$; and first the intersections of a partial branch $z-x^{\frac{m_1}{m_1-\mu_1}}$ by each of the remaining $n_1(m_1-\mu_1)-1$ partial branches of the same set: the number of intersections with any one of these is $=\frac{m_1}{m_1-\mu_1}$; and con-

sequently the number with all of them is $=\frac{m_1}{m_1-\mu_1}\,[n_1(m_1-\mu_1)-1]$. But we obtain this same number from each of the $n_1(m_1-\mu_1)$ partial branches, and thus the whole number is $n_1(m_1-\mu_1)\,\frac{m_1}{m_1-\mu_1}\,[n_1(m_1-\mu_1)-1]$, $=n_1m_1[n_1(m_1-\mu_1)-1]$.

Taking account of the other sets, each with itself, the whole number of such intersections is

$$n_1m_1[n_1(m_1-\mu_1)-1]+n_2m_2[n_2(m_2-\mu_2)-1]+n_3m_3[n_3(m_3-\mu_3)-1],$$

which is

$$=\Sigma' n^2m^2-\Sigma' n^2m\mu-\Sigma' nm.$$

Observe now that $\frac{m_1}{\mu_1}>\frac{m_2}{\mu_2}$, that is $\frac{\mu_1}{m_1}<\frac{\mu_2}{m_2}$, and that, these being each <1, we thence have $1-\frac{\mu_1}{m_1}>1-\frac{\mu_2}{m_2}$, that is $\frac{m_1-\mu_1}{m_1}>\frac{m_2-\mu_2}{m_2}$: and we thus have

$$\frac{m_1}{m_1-\mu_1}<\frac{m_2}{m_2-\mu_2}<\frac{m_3}{m_3-\mu_3}.$$

Considering now the intersections of partial branches of the two sets $\left(z-x^{\frac{m_1}{m_1-\mu_1}}\right)^{n_1(m_1-\mu_1)}$ and $\left(z-x^{\frac{m_2}{m_2-\mu_2}}\right)^{n_2(m_2-\mu_2)}$ respectively, a partial branch $z-x^{\frac{m_1}{m_1-\mu_1}}$ gives with each partial branch of the other set a number $=\frac{m_1}{m_1-\mu_1}$; and in this way taking each partial branch of each set, the number is $n_1(m_1-\mu_1).n_2(m_2-\mu_2).\frac{m_1}{m_1-\mu_1}$, $=n_1m_1n_2(m_2-\mu_2)$; and thus for all the sets the number is

$$=n_1m_1n_2(m_2-\mu_2)+n_1m_1n_3(m_3-\mu_3)+n_2m_2n_3(m_3-\mu_3),$$

which is

$$=\Sigma' n_rm_rn_sm_s-\sum_{s>r}\Sigma' n_rm_rn_s\mu_s,$$

where in the first sum the Σ' refers to each pair of values of the suffixes. But the intersections are to be taken twice; the number thus is

$$=2\Sigma' n_rm_rn_sm_s-2\sum_{s>r}\Sigma n_rm_rn_s\mu_s.$$

Adding the foregoing number

$$\Sigma' n^2m^2-\Sigma' n^2m\mu-\Sigma' nm,$$

the whole number for the singularity in question is

$$=(\Sigma' nm)^2-\Sigma' nm-\Sigma' n^2m\mu-2\sum_{s>r}\Sigma' n_rm_rn_s\mu_s.$$

FRANKLIN INSTITUTE LIBRARY

Similarly for the singularity $\left(z-y^{\frac{\mu_5}{\mu_5-m_5}}\right)^{n_5(\mu_5-m_5)}$ . . . ; taking each set with itself, the number of intersections is

$$n_5\mu_5[n_5(\mu_5-m_5)-1]+n_6\mu_6[n_6(\mu_6-m_6)-1]+n_7\mu_7[n_7(\mu_7-m_7)-1],$$

which is

$$=\Sigma''n^2\mu^2-\Sigma''n^2m\mu-\Sigma''n\mu.$$

We have here $\frac{m_5}{\mu_5}>\frac{m_6}{\mu_6}$ and each of these being less than 1, we have $1-\frac{m_5}{\mu_5}<1-\frac{m_6}{\mu_6}$, that is $\frac{\mu_5-m_5}{\mu_5}<\frac{\mu_6-m_6}{\mu_6}$, or $\frac{\mu_5}{\mu_5-m_5}>\frac{\mu_6}{\mu_6-m_6}$; and so

$$\frac{\mu_7}{\mu_7-m_7}<\frac{\mu_6}{\mu_6-m_6}<\frac{\mu_5}{\mu_5-m_5}.$$

Hence considering the two sets $\left(z-y^{\frac{\mu_5}{\mu_5-m_5}}\right)^{n_5(\mu_5-m_5)}$ and $\left(z-y^{\frac{\mu_6}{\mu_6-m_6}}\right)^{n_6(\mu_6-m_6)}$, a partial branch of the first set gives with a partial branch of the second set $\frac{\mu_6}{\mu_6-m_6}$ intersections: and the number thus obtained is $n_5(\mu_5-m_5).n_6(\mu_6-m_6).\frac{\mu_6}{\mu_6-m_6}, =n_5n_6\mu_6(\mu_5-m_5)$.

For all the sets the number is

$$n_5n_6\mu_6(\mu_5-m_5)+n_5n_7\mu_7(\mu_5-m_5)+n_6n_7\mu_7(\mu_6-m_6)$$

or taking this twice, the number is

$$=2\Sigma''n_rn_s\mu_r\mu_s-2\sum_{r>s}''n_rm_sn_s\mu_s$$

where in the first sum the Σ'' refers to each pair of suffixes. Adding the foregoing value

$$\Sigma''n^2\mu^2-\Sigma''n^2m\mu-\Sigma''n\mu,$$

the whole number for the singularity in question is

$$=(\Sigma''n\mu)^2-\Sigma''n\mu-\Sigma''n^2m\mu-2\sum_{r>s}''n_rm_sn_s\mu_s;$$

and the proof is thus completed.

Referring to the foot-note ante (p. 753), I remark that the theorem $\gamma=$ deficiency, is absolute, and applies to a curve with any singularities whatever: in a curve which has singularities not taken account of in ABEL's theory, the "quelques cas particuliers que je me dispense de considérer," the singularities not taken account of give rise to a diminution in the deficiency of the curve, and also to an equal diminution of the value of γ as determined by ABEL's formula; and the actual deficiency will be = ABEL's γ — such diminution, that is, it will be = true value of γ.

XVIII. *On* Riccati's *Equation and its Transformations, and on some Definite Integrals which satisfy them.*

By J. W. L. Glaisher, *M.A., F.R.S., Fellow of Trinity College, Cambridge.*

Received and read June 16, 1881.

Introduction.

The present memoir relates chiefly to the different forms of the particular integrals of the differential equation

$$\frac{d^2u}{dx^2}-a^2u=\frac{p(p+1)}{x^2}u \quad \ldots \quad (1),$$

and to the evaluation of certain definite integrals which are connected with this equation. Transforming (1) by assuming $u=x^{-p}v$, it becomes

$$\frac{d^2v}{dx^2}-\frac{2p}{x}\frac{dv}{dx}-a^2v=0 \quad \ldots \quad (2),$$

that is, writing $n-1$ for $2p$,

$$\frac{d^2v}{dx^2}-\frac{n-1}{x}\frac{dv}{dx}-a^2v=0 \quad \ldots \quad (3),$$

and this equation may be transformed into

$$\frac{d^2v}{dz^2}-a^2z^{2q-2}v=0 \quad \ldots \quad (4),$$

by the substitution $x=\frac{1}{q}z^q$, where $q=\frac{1}{n}$. The equation (4) may be regarded as the standard form of Riccati's equation (see § III., art. 17).

It is well-known that these equations admit of integration in a finite form if $p=$ an integer, $n=$ an uneven integer, and $q=$ the reciprocal of an uneven integer, respectively.

The contents of the memoir are as follows:

In the first section (§ I.) six particular integrals of the equation (1) are obtained, and the relations between them are examined. When p is not an integer, all the six integrals extend to infinity, and in this case the relations between them present no special peculiarity. When p is an integer, two of the series terminate, and we thus obtain two particular integrals of (1) which contain a finite number of terms. The

series terminate in consequence of the occurrence of zero factors in the coefficients of the terms, but if they are continued, zero factors occur also in the denominators, so that, after a finite number of zero terms, the series may be regarded as recommencing and extending to infinity. If the terminating series are supposed to recommence in this manner, so that all the series extend to infinity, then the relations between the particular integrals are the same as when p is not an integer; but if the series are supposed to terminate absolutely when the zero terms occur, the relations are quite different. As the finite portions of the particular integrals satisfy the differential equation, it is more natural to regard the series as terminating absolutely, and on this supposition the relations between the particular integrals exhibit a remarkable diversity of form according as p is or is not an integer.

The second section contains what is believed to be a new form of the solution of (1) in the case of $p =$ an integer. It is shown that if $p=i$, a positive integer, this equation is satisfied by the coefficient of h^{i+1} in the expansion of $e^{x\sqrt{(x^2+xh)}}$ in ascending powers of h. The six particular integrals given in § I. of the equation (1) and the relations connecting them are obtained by different expansions of this expression.

The third section contains the six particular integrals of (3) and (4) corresponding to those of (1), from which they are deduced by means of the transformations stated above.

The fourth section relates to the particular cases in which the differential equations admit of integration in a finite form. If a differential equation is satisfied by an infinite series, and if for certain values of a quantity involved in it the series terminates, then in this case we may present the integral in a different form by commencing the finite series at the other end, and writing the terms in the reverse order.

Thus, for example, a particular integral of (1) is $u=\mathrm{P}$, where

$$\mathrm{P}=x^{-p}\left\{1-\frac{p}{p}ax+\frac{p(p-1)}{p(p-\frac{1}{2})}\frac{a^2x^2}{2!}-\frac{p(p-1)(p-2)}{p(p-\frac{1}{2})(p-1)}\frac{a^3x^3}{3!}+\&c.\right\}e^{ax},$$

but, if $p =$ a positive integer, then commencing the series at the other end,

$$\mathrm{P}=(-)^p\frac{2^pa^p}{(p+1)(p+2)\ldots 2p}\left\{1-\frac{p(p+1)}{2}\frac{1}{ax}+\frac{(p-1)p(p+1)(p+2)}{2.4}\frac{1}{a^2x^2}\cdots\right.$$
$$\left.+(-)^p\frac{1.2\ldots 2p}{2.4\ldots 2p}\frac{1}{a^px^p}\right\}e^{ax}.$$

These reverse forms in the case of the equations (1), (3), (4) are given in this section.

It is worthy of remark that if we are given a particular integral of a differential equation in the form of a terminating series, such as, for example,

$$1-\frac{p(p+1)}{2}\frac{1}{ax}+\frac{(p-1)p(p+1)(p+2)}{2.4}\frac{1}{a^2x^2}-\&c.,$$

p being a positive integer, then we might suppose that the corresponding particular integral, when p was not an integer, would be obtained by continuing the series, which does not then terminate, to infinity. This infinite series, when p is not an integer, still satisfies the differential equation, but is divergent; and the true integral is obtained by commencing the series at the other end and continuing it to infinity backwards. In general, when we have a series which terminates of itself for a particular form of p, we may derive from it two infinite series, when p has not this form, by commencing it at either end. One of these will be an ascending series and the other a descending series; and we can thus, as it were, pass from the one to the other through the intervention of the finite series.

The fifth section contains the evaluations of the definite integrals

$$\int_0^\infty e^{-x^m-\frac{a^2}{x^m}}dx, \qquad \int_0^\infty \frac{\cos bx}{(a^2+x^2)^n}dx,$$

where m denotes any real quantity and n any positive quantity. These integrals have been evaluated when m is of the form $\frac{-4i}{2i\pm1}$ and when n is a positive integer; but, so far as I know, the general formulæ given are new. It is known that these integrals satisfy differential equations of the forms (4) and (1) respectively, so that their values are necessarily connected with the solutions of these equations considered in §§ I. and III. The results are curious, as they exhibit changes of form similar to those referred to in describing the contents of § I., and which are due to the same cause—viz. the recommencement of the terminating series after the zero terms.

When n is unrestricted it is shown that we have

$$\int_0^\infty x^{m-1}e^{-x^m-\frac{a^2}{x^m}}dx= \quad \tfrac{1}{2}\Gamma(\tfrac{1}{2}n)\left\{1+\frac{n-1}{n-1}(2a)+\frac{(n-1)(n-3)}{(n-1)(n-2)}\frac{(2a)^2}{2!}+\&c.\right\}e^{-2a}$$

$$+\tfrac{1}{2}\Gamma(-\tfrac{1}{2}n)a^n\left\{1+\frac{n+1}{n+1}(2a)+\frac{(n+1)(n+3)}{(n+1)(n+2)}\frac{(2a)^2}{2!}+\&c.\right\}e^{-2a};$$

but when n is a positive integer the first series is to be continued till it terminates, and the second is to be ignored; and if n is a negative integer the second is to be continued till it terminates, and the first is to be ignored. The well-known value of the integral when $n-1=$ an even integer $=2i$, viz.

$$\int_0^\infty x^{2i}e^{-x^2-\frac{a^2}{x^2}}dx=\frac{\sqrt{\pi}}{2}a^i\left\{1+\frac{i(i+1)}{2}\frac{1}{2a}+\frac{(i-1)i(i+1)(i+2)}{2.4}\left(\frac{1}{2a}\right)^2+\&c.\right\}e^{-2a},$$

does not suggest the general formula, the terms of the finite series being written in the reverse order.

Certain formulæ of BOOLE's and CAUCHY's are also considered and extended in this section.

The sixth section, which is the longest in the memoir, relates to the numerous symbolic solutions of the equation (1) and its transformations (3) and (4) in the cases in which they are integrable in finite terms. In this section these symbolic solutions are derived from the definite integrals considered in § V.; and the various symbolic theorems to which they lead by comparing different forms of the results are examined. A great many symbolic solutions of the differential equations have been given by R. L. ELLIS, BOOLE, LEBESGUE, HARGREAVE, WILLIAMSON, DONKIN, &c., and these are briefly noticed and connected with one another. It may be observed that the solution

$$u=x^{p+1}\left(\frac{1}{x}\frac{d}{dx}\right)^{p+1}\left(\frac{c_1e^{ax}+c_2e^{-ax}}{x}\right),$$

which has been several times independently discovered, seems to have been first published by Mr. GASKIN, who in effect gave it in a problem set in the Senate House Examination at Cambridge in 1839.

The seventh section relates to the connexion between the results given in §§ I.–VI. and the formulæ of BESSEL's Functions. BESSEL's equation

$$\frac{d^2w}{dx^2}+\frac{1}{x}\frac{dw}{dx}+\left(1-\frac{\nu^2}{x^2}\right)w=0,$$

may be derived from (1) by the simple substitutions

$$u=x^kw, \quad p+\tfrac{1}{2}=\nu, \quad a^2=-1;$$

so that all the theorems relating to the solutions of (1) have analogues in the solutions of BESSEL's equation, which are deducible from them by these transformations. In this section the formulæ in BESSEL's Functions which correspond to those considered in the memoir are stated in a convenient form for comparison. The number of such formulæ is not great, and the substitution of $\sqrt{(-1)}$ for a, which converts exponentials into sines and cosines, and a single series multiplied by an exponential factor into the sum of two series multiplied respectively by a sine and a cosine, changes considerably the appearance of the results, which, from an analytical point of view, are less simple when the differential equation is of BESSEL's form. The principal case considered in the theory of BESSEL's Functions is that of ν = an integer: this corresponds to the case of p = an integer $+\frac{1}{2}$, which is generally excluded in this memoir, as it renders certain of the particular integrals infinite (§ I., arts. 1, 3). The case of

finite solution corresponds in (1) to p = an integer and in BESSEL's equation to ν = an integer $+\frac{1}{2}$. The fact that BESSEL's function $J^{\nu}(x)$ is expressible in a finite form when $\nu = i + \frac{1}{2}$, and the finite expression itself, are well known, and the case is an important one in physical investigations; but, so far as I know, the recommencement of the series after the zero terms has not been specially noticed in connexion with the subject of BESSEL's Functions.

The eighth (and last) section contains a list of writings the contents of which are closely connected with the subject of the memoir, arranged in order of date and classed under the sections in which they are noticed. There is also in each case a short account of the portion of the paper for which it has been referred to, with the numbers of the articles in which the references occur. The section does not contain a list of all the papers referred to in the memoir; only those papers which are closely connected with it, and portions of which are, in most cases, to some extent reproduced in it, being included. The part of the list which relates to § VI. is intended to be supplementary to that section: it is not in any sense a bibliography of the symbolic solutions, but it probably contains references to all the more important papers on the subject.

In the 'Philosophical Magazine' for 1868 CAYLEY gave the four particular integrals P_2, Q_2, R_2, S_2 (§ III.) of RICCATI's equation (4); and in the same journal for 1872 I investigated the relations between these four particular integrals and the well-known particular integrals U_2, V_2. The results are the same as those given in § III., and the method is similar to that employed in § I. I afterwards found that the process of obtaining and connecting the particular integrals assumed a much more simple form when the differential equation was taken to be (1) than when it was (4); and it seemed desirable to re-write the whole investigation, taking (1) as the differential equation. This investigation forms § I.; it is similar in every respect to that contained in the 'Philosophical Magazine,' but is much more complete. The corresponding results for the equations (3) and (4) are deduced in § III.

The fact that, in the solution in series of a differential equation, if the series terminates but when continued recommences, the latter portion as well as the finite series satisfies the differential equation, was pointed out by CAYLEY in the 'Messenger of Mathematics' for 1869.

The formula (8) of § V. was published in the 'British Association Report' for 1872, with a brief account of the process given in arts. 20, 21. The principal portion of two short papers, "On RICCATI's Equation" and "On certain Differential Equations allied to RICCATI's," which were published in the 'Quarterly Journal of Mathematics' for 1871 and 1872, are incorporated in § VI.

The memoir thus includes the results contained in several scattered notes and papers. In these the differential equation considered was generally RICCATI's in the form (4), but the advantage of adopting (1) as the standard form in preference to (4) is considerable. As far as the differential equation is concerned (4), which consists of

only two terms, is the simplest form; but as regards the expression of the results, both (1) and (3) are superior in every respect. The equation (3) was adopted as the standard form by M. BACH in his paper of 1874 (see § VIII.).

The form $2q-2$ for the exponent in RICCATI'S equation (4) was first employed, I believe, in CAYLEY'S paper in the 'Philosophical Magazine' for 1868, which has been already referred to. The use of the quantity q greatly simplifies the formulæ relating to the solution of the equation.

With the exception of § VII., the memoir was written about three years ago, the delay in communicating it to the Society being due to the fact that it seemed desirable to connect the results more closely with BESSEL'S Functions. As the theory of these functions forms a distinct and recognised branch of analysis, and as the differential equations considered are transformable into BESSEL'S equation by very simple changes in the variables, it was clearly of importance to examine with some care the connexion of the formulæ with those of BESSEL'S Functions, and it even seemed possible that it might be advisable to adopt BESSEL'S equation as the standard form. For the reasons already stated it appeared that this was not the case, and that the analytical treatment of the subject was complicated by the change to BESSEL'S equation. It is well known that the general integrals of the differential equations (1), ... (4) can be expressed in terms of BESSEL'S Functions; and LOMMEL has specially considered these solutions in several papers in the 'Mathematische Annalen.'* In these papers, however, the points to which the memoir relates are not referred to. It therefore seemed sufficient to give in § VII. the connexion between the principal formulæ, reserving for a separate paper, if it should appear desirable, the examination of the relations in which the series considered in the memoir stand to BESSEL'S Functions with negative indexes and to the functions of the second kind introduced by LOMMEL and by NEUMANN.

During the time that the memoir has been in manuscript I have published two extracts from it, viz. the theorem in § II., arts. 8, 9, in the 'British Association Report' for 1880, and the theorem (50) and its proof (§ VI., art. 41) in the 'Proceedings of the Cambridge Philosophical Society' for 1879.

The differential equations (1), ... (4) present three distinct peculiarities, viz. (i.) they are finitely integrable only in special cases; (ii.) they are satisfied by certain remarkable definite integrals, which have attracted attention quite independently of the differential equations; and (iii.) the solutions when finite admit of being exhibited in various symbolic forms. In reference to the third of these properties, it is remarkable how much attention has been devoted to the solutions of the equations in these finite cases during the last fifty years. The differential equations (1), ... (4) have however been frequently discussed not as simple transformations of one another

* Vol. ii. (1870), pp. 624–635; vol. iii. (1871), pp. 475–487; vol. xiv. (1879), pp. 510–536.

but as if they were essentially different, and the processes of solution have been applied to them independently. Also many of the forms have been re-discovered several times; and it therefore seemed to be worth while to collect together, as in § VI., the different symbolic formulæ, and exhibit the nature of the relations between them.

Although the equation (1) is connected with BESSEL's equation by such simple relations, the methods of treatment of the two equations by mathematicians have been very different. In the case of (1) and its transformations (2), (3), (4), the purely analytical part of the theory and the forms of the solutions have chiefly attracted attention; while in the case of BESSEL's equation the expansion of the results in series suitable for calculation has been one of the main objects. The theories of the two equations have been developed from very different points of view; the one has been considered in reference to the methods of solution and the peculiarities already referred to, and the other has been considered almost wholly in connexion with the functions which satisfy it, and their applications in astronomy and physics. It is curious that two such very distinct classes of analytical investigation should have been formed having reference to differential equations so closely connected.

It is proper to remark here that in the differential equation (1) and throughout the memoir the constant a may be put equal to unity without loss of generality. It was found to be desirable to retain it, as there is some advantage in having present in the solutions a letter whose sign can be changed at pleasure, and also because the transition to the differential equations

$$\frac{d^2u}{dx^2}+a^2u=\frac{p(p+1)}{x^2}u, \quad \&c.,$$

(*i.e.*, in which the sign of a^2 is changed) is thus rendered somewhat more convenient.

The ordinary differential equations (1), ... (4) are considered throughout, and no reference is made to the corresponding partial differential equations

$$\frac{d^2u}{dx^2}-a^2\frac{d^2u}{dy^2}=\frac{p(p+1)}{x^2}u, \quad \&c.,$$

the solutions of which may be deduced in the usual manner by replacing a by $a\frac{d}{dy}$, and c_1e^{ax} and c_2e^{-ax} by $\phi(y+ax)$ and $\psi(y+ax)$. No point of interest arises in connexion with this transition.

Following the notation usually adopted in connexion with the differential equation (1), i is used throughout to denote a positive integer. The expression $\sqrt{(-1)}$, which occurs only towards the end of § VI. and in § VII., is denoted by i.

The headings of the eight sections, with the numbers of the articles which they contain and the pages, are as follows:

§ I. Direct integration of the differential equation in series, and connexion between the particular integrals. Arts. 1–7; pp. 766–774.
§ II. Integration of the differential equation when p = an integer. Arts. 8–15; pp. 774–779.
§ III. Transformations of the original differential equation. RICCATI'S equation. Arts. 16, 17; pp. 779–782.
§ IV. Special forms of the particular integrals in the cases in which the differential equations admit of integration in a finite form. Arts. 18, 19; pp. 783, 784.
§ V. Evaluation of definite integrals satisfying the differential equations. Arts. 20–28, pp. 784–797.
§ VI. Symbolic forms of the particular integrals in the cases in which the differential equations admit of integration in a finite form. Arts. 29–42; pp. 798–819.
§ VII. Connexion with BESSEL's Functions. Arts. 43–48; pp. 819–822.
§ VIII. Writings specially connected with the contents of the memoir. Pp. 823–828.

§ I.

Direct integration of the differential equation in series, and connexion between the particular integrals. Arts. 1–7.

1. The most direct method of integrating the differential equation

$$\frac{d^2u}{dx^2}-a^2u=\frac{p(p+1)}{x^2}u \quad \ldots\ldots\ldots\ldots \quad (1),$$

and obtaining the relations that exist between the different particular integrals, appears to be as follows.

Let

$$u=\Sigma A_r x^{m+r},$$

the summation extending to all positive integral values of r; then, substituting in the differential equation, we have

$$(m+r+p)(m+r-p-1)A_r-a^2A_{r-2}=0,$$

whence, putting $r=0$ or 1,

$$m=-p \text{ or } p+1.$$

Taking the first root, the equations giving A_2, A_4, A_6 ... are

$$\begin{aligned} 2(1-2p)A_2-a^2A_0&=0,\\ 4(3-2p)A_4-a^2A_2&=0,\\ 6(5-2p)A_6-a^2A_4&=0,\\ \ldots\ldots\ldots& \end{aligned}$$

whence

$$A_2 = -\frac{1}{p-\frac{1}{2}}\frac{a^2}{2^2}A_0,$$

$$A_4 = -\frac{1}{2}\frac{1}{p-\frac{3}{2}}\frac{a^2}{2^2}A_2,$$

$$A_6 = -\frac{1}{3}\frac{1}{p-\frac{5}{2}}\frac{a^2}{2^2}A_4,$$

. . . . . .

so that the solution corresponding to the root $m=-p$ is

$$U = x^{-p}\left\{1 - \frac{1}{p-\frac{1}{2}}\frac{a^2x^2}{2^2} + \frac{1}{(p-\frac{1}{2})(p-\frac{3}{2})}\frac{a^4x^4}{2^4.2!} - \frac{1}{(p-\frac{1}{2})(p-\frac{3}{2})(p-\frac{5}{2})}\frac{a^6x^6}{2^6.3!} + \&c.\right\},$$

where, as throughout this memoir, $r!$ denotes $1.2.3 \ldots r$.

Similarly, taking the root $m=-p-1$, the other solution is found to be

$$V = x^{p+1}\left\{1 + \frac{1}{p+\frac{3}{2}}\frac{a^2x^2}{2^2} + \frac{1}{(p+\frac{3}{2})(p+\frac{5}{2})}\frac{a^4x^4}{2^4.2!} + \frac{1}{(p+\frac{3}{2})(p+\frac{5}{2})(p+\frac{7}{2})}\frac{a^6x^6}{2^6.3!} + \&c.\right\},$$

and, as U and V are independent, the complete integral of the differential equation is $AU+BV$, A and B being arbitrary constants.

There is nothing in the form of these series to indicate that for any values of p the integral of the differential equation admits of being expressed in a finite form. They show, however, that if p = the half of an uneven integer (the case $p=-\frac{1}{2}$ alone excepted) the solution assumes a different form, viz. if, say, in U the terms after a certain point become infinite, the solution is of the form $W+V\log cx$, W being a new series. This case is excluded in what follows; and throughout the memoir p is supposed not to be of the forms $\pm\frac{1}{2}(2n+1)$. If, however, p is of either of these forms only certain of the series considered will involve infinite terms, and the relations connecting those series which do not involve infinite terms will still remain true.

2. Transforming the differential equation (1) by assuming $u=e^{ax}v$, a substitution suggested by the form of the first member of the equation, we obtain the differential equation in v

$$\frac{d^2v}{dx^2} + 2a\frac{dv}{dx} = \frac{p(p+1)}{x^2}v.$$

Putting as before

$$v = \Sigma A_r x^{m+r},$$

we have

$$(m+r+p)(m+r-p-1)A_r + 2(m+r-1)aA_{r-1} = 0,$$

whence

$$m = -p \text{ or } p+1.$$

Taking the first root, the equations are

$$\begin{aligned}(-2p)A_1+2(-p)aA_0&=0,\\ 2(1-2p)A_2+2(1-p)aA_1&=0,\\ 3(2-2p)A_3+2(2-p)aA_2&=0,\\ &\cdots\cdots\cdots\cdots\end{aligned}$$

giving

$$A_1=-\frac{p}{p}aA_0,\qquad A_2=-\tfrac{1}{2}\frac{p-1}{p-\frac{1}{2}}aA_1,\qquad A_3=-\tfrac{1}{3}\frac{p-2}{p-1}aA_2,\ \ldots\ ;$$

and we obtain the particular integral

$$x^{-p}\left\{1-\frac{p}{p}ax+\frac{p(p-1)}{p(p-\frac{1}{2})}\frac{a^2x^2}{2!}-\frac{p(p-1)(p-2)}{p(p-\frac{1}{2})(p-1)}\frac{a^3x^3}{3!}+\&c.\right\}.$$

Similarly, the other particular integral is found to be

$$x^{p+1}\left\{1+\frac{p+1}{p+1}ax+\frac{(p+1)(p+2)}{(p+1)(p+\frac{3}{2})}\frac{a^2x^2}{2!}+\frac{(p+1)(p+2)(p+3)}{(p+1)(p+\frac{3}{2})(p+2)}\frac{a^3x^3}{3!}+\&c.\right\}.$$

If we had transformed (1) by assuming $u=e^{-ax}v$, we should have obtained a differential equation in v differing from that given above only in having the sign of a changed: and the two particular integrals would differ from those just written only in having the signs of the alternate terms negative.

3. Thus, of the differential equation

$$\frac{d^2u}{dx^2}-a^2u=\frac{p(p+1)}{x^2}u,$$

we have obtained the six particular integrals U, V, P, Q, R, S, where

$$U=x^{-p}\left\{1-\frac{1}{p-\frac{1}{2}}\frac{a^2x^2}{2^2}+\frac{1}{(p-\frac{1}{2})(p-\frac{3}{2})}\frac{a^4x^4}{2^4.2!}-\frac{1}{(p-\frac{1}{2})(p-\frac{3}{2})(p-\frac{5}{2})}\frac{a^6x^6}{2^6.3!}+\&c.\right\},$$

$$V=x^{p+1}\left\{1+\frac{1}{p+\frac{3}{2}}\frac{a^2x^2}{2^2}+\frac{1}{(p+\frac{3}{2})(p+\frac{5}{2})}\frac{a^4x^4}{2^4.2!}+\frac{1}{(p+\frac{3}{2})(p+\frac{5}{2})(p+\frac{7}{2})}\frac{a^6x^6}{2^6.3!}+\&c.\right\},$$

$$P=x^{-p}\left\{1-\frac{p}{p}ax+\frac{p(p-1)}{p(p-\frac{1}{2})}\frac{a^2x^2}{2!}-\frac{p(p-1)(p-2)}{p(p-\frac{1}{2})(p-1)}\frac{a^3x^3}{3!}+\&c.\right\}e^{ax},$$

$$Q=x^{p+1}\left\{1-\frac{p+1}{p+1}ax+\frac{(p+1)(p+2)}{(p+1)(p+\frac{3}{2})}\frac{a^2x^2}{2!}-\frac{(p+1)(p+2)(p+3)}{(p+1)(p+\frac{3}{2})(p+2)}\frac{a^3x^3}{3!}+\&c.\right\}e^{ax},$$

$$R=x^{-p}\left\{1+\frac{p}{p}ax+\frac{p(p-1)}{p(p-\frac{1}{2})}\frac{a^2x^2}{2!}+\frac{p(p-1)(p-2)}{p(p-\frac{1}{2})(p-1)}\frac{a^3x^3}{3!}+\&c.\right\}e^{-ax},$$

$$S=x^{p+1}\left\{1+\frac{p+1}{p+1}ax+\frac{(p+1)(p+2)}{(p+1)(p+\frac{3}{2})}\frac{a^2x^2}{2!}+\frac{(p+1)(p+2)(p+3)}{(p+1)(p+\frac{3}{2})(p+2)}\frac{a^3x^3}{3!}+\&c.\right\}e^{-ax}.$$

These integrals form three pairs U and V, P and Q, R and S, either of the integrals in each pair being deducible from the other by the substitution of $-(p+1)$ for p: and, since the differential equation involves p only in the form $p(p+1)$, it is evident *à priori* that if in any expression satisfying the differential equation, p is replaced by $-(p+1)$, the new expression must still satisfy the differential equation.

Also the pairs P and Q, R and S, are convertible the one into the other by changing the sign of α.

4. If p is a positive integer the series in P and R terminate and the general integral of the differential equation is AP+BR; and if p is a negative integer, the series in Q and S terminate and the general integral is AQ+BS.

Thus, if $p=2$, the general integral is

$$u=Ax^{-2}\{1-\alpha x+\tfrac{1}{3}\alpha^2x^2\}e^{\alpha x}+Bx^{-2}\{1+\alpha x+\tfrac{1}{3}\alpha^2x^2\}e^{-\alpha x};$$

and, if $p=-2$, the general integral is

$$u=Ax^{-1}\{1-\alpha x\}e^{\alpha x}+Bx^{-1}\{1+\alpha x\}e^{-\alpha x}.$$

5. As however we have six particular integrals, of which, for any given value of p, only two can be independent, it remains to investigate the relations between the particular integrals in the different cases that arise.

(1°.) Suppose p unrestricted (except as mentioned in art. 1), but not equal to an integer.

In this case all the series extend to infinity, and

$$P=R=U, \qquad Q=S=V$$

for, leaving out of consideration the factor x^{-p} that occurs in both P and U, the coefficient of α^nx^n in P

$$=\frac{1}{n!}-\frac{p}{p}\frac{1}{(n-1)!}+\frac{p(p-1)}{p(p-\frac{1}{2})}\frac{1}{(n-2)!2!}\cdots+(-)^n\frac{p(p-1)\ldots\{p-(n-1)\}}{p(p-\frac{1}{2})\ldots\{p-\frac{1}{2}(n-1)\}}\frac{1}{n!}$$

$$=\frac{1}{n!}-\frac{p}{2p}\frac{2}{(n-1)!}+\frac{p(p-1)}{2p(2p-1)}\frac{2^2}{(n-2)!2!}\cdots+(-)^n\frac{p(p-1)\ldots\{p-(n-1)\}}{2p(2p-1)\ldots\{2p-(n-1)\}}\frac{2^n}{n!}$$

$$=\frac{1}{2p(2p-1)\ldots\{2p-(n-1)\}}\left\{\frac{2p(2p-1)\ldots\{2p-(n-1)\}}{n!}\right.$$

$$\left.-\frac{(2p-1)(2p-2)\ldots\{2p-(n-1)\}}{(n-1)!}p.2\ldots+(-)^n\frac{p(p-1)\ldots\{p-(n-1)\}}{n!}2^n\right\},$$

and we see that the expression in brackets is equal to the coefficient of t^n in the expansion of

$$(1+t)^{2p}-p.2t(1+t)^{2p-1}+\frac{p(p-1)}{2!}2^2t^2(1+t)^{2p-2}\ldots$$

$$+(-)^n\frac{p(p-1)\ldots\{p-(n-1)\}}{n!}2^nt^n(1+t)^{2p-n},$$

that is, in

$$(1+t)^{2p}\left(1-\frac{2t}{1+t}\right)^p=(1+t)^p(1-t)^p=(1-t^2)^p.$$

If, therefore, n is uneven the coefficient of a^nx^n in P is zero, and if n is even the coefficient

$$=\frac{1}{2p(2p-1)\ldots\{2p-(n-1)\}}\times(-)^{\frac{1}{2}n}\frac{p(p-1)\ldots\{p-\frac{1}{2}n+1)\}}{(\frac{1}{2}n)!}$$

$$=(-)^{\frac{1}{2}n}\frac{1}{(p-\frac{1}{2})(p-\frac{2}{2})\ldots\{p-\frac{1}{2}(n-1)\}}\frac{1}{2^{\frac{1}{2}n}(\frac{1}{2}n)!},$$

which is the coefficient of a^nx^n in U.

Since R differs from P only in having the sign of a changed, and since U is a function of a^2 only, it follows that P=R=U. Also, since Q, S, V differ from P, R, U only in having $-(p+1)$ in place of p, it follows that Q=S=V.

(2°). Suppose p a positive integer, $=i$, say.

In this case the $(i+1)^{\text{th}}$ term of the series in P, including the factor x^{-i}, is

$$x^{-i}.(-)^i\frac{i(i-1)(i-2)\ldots\{i-(i-1)\}}{i(i-\frac{1}{2})(i-1)\ldots\{i-\frac{1}{2}(i-1)\}}\frac{a^ix^i}{i!},$$

and the next term vanishes owing to the presence of the factor $i-i$ or 0 in the numerator.

For the same reason all the succeeding terms vanish until the factor $i-i$ appears in the denominator also, when the zero factors cancel one another and the series recommences, the first term of the new series being

$$x^{-i}.-\frac{i(i-1)\ldots1.0.-1.-2\ldots(i-2i)}{i(i-\frac{1}{2})\ldots\{i-\frac{1}{2}(2i-1)\}.0}\frac{a^{2i+1}x^{2i+1}}{(2i+1)!}$$

$$=(-)^{i+1}\left(\frac{i!}{2i!}\right)^2\frac{1}{2i+1}\frac{1}{2^{2i}}a^{2i+1}x^{i+1}$$

$$=(-)^{i+1}\frac{2i+1}{\{1.3.5\ldots(2i+1)\}^2}a^{2i+1}x^{i+1}$$

$$=gx^{i+1},\text{ where }g=(-)^{i+1}\frac{2i+1}{\{1.3.5\ldots(2i+1)\}^2}a^{2i+1}.$$

The new series, multiplied by the factor e^{ax}, thus becomes

$$gx^{i+1}\left\{1-\frac{i+1}{i+1}ax+\frac{(i+1)(i+2)}{(i+1)(i+\frac{3}{2})}\frac{a^2x^2}{2!}-\&c.\right\}e^{ax}$$
$$=gQ.$$

Denoting then by P′ the finite part of P, the series being supposed to end at the term immediately preceding the first term which contains a zero factor in the numerator, viz. putting

$$P'=x^{-i}\left\{1-\frac{i}{i}ax+\frac{i(i-1)}{i(i-\frac{1}{2})}\frac{a^2x^2}{2!}\cdots+(-)^i\frac{i!}{i(i-\frac{1}{2})\ldots\{i-\frac{1}{2}(i-1)\}}\frac{a^ix^i}{i!}\right\}e^{ax},$$

we have found that

$$P=P'+gQ=U.$$

Similarly, if R′ denotes the finite part of R, the series ending at the term immediately preceding the first term which contains a zero factor in the numerator, we find that

$$R=R'-gS=U,$$

and also, as before,

$$Q=S=V.$$

The proof in (1°) that P=U does not apply as it stands when $p=i$, but it can be extended so as to include this case by putting $p=i+h$, and making h indefinitely small. The equality of P and U for all values of p may however be proved without the use of limits by showing that the coefficient of x^n in Ue^{-ax} is equal to the coefficient of x^n in P. To prove this; first suppose n to be even and $=2m$, then the coefficient of $a^{2m}x^{2m}$ in x^pUe^{-ax} is equal to

$$\frac{1}{(2m)!}-\frac{1}{p-\frac{1}{2}}\frac{1}{2^2}\frac{1}{(2m-2)!}+\frac{1}{(p-\frac{1}{2})(p-\frac{3}{2})}\frac{1}{2^4.2!}\frac{1}{(2m-4)!}\cdots$$
$$+(-)^m\frac{1}{(p-\frac{1}{2})(p-\frac{3}{2})\ldots(p-m+\frac{1}{2})}\frac{1}{2^{2m}.m!}$$
$$=\frac{1}{2m(2m-1)\ldots(m+1)}\frac{1}{(p-\frac{1}{2})(p-\frac{3}{2})\ldots(p-m+\frac{1}{2})}\left\{\frac{(p-\frac{1}{2})(p-\frac{3}{2})\ldots(p-m+\frac{1}{2})}{m!}\right.$$
$$\left.-(m-\frac{1}{2})\frac{(p-\frac{3}{2})\ldots(p-m+\frac{1}{2})}{(m-1)!}\cdots+(-)^m\frac{2m(2m-1)\ldots(m+1)}{m!}\frac{1}{2^{2m}}\right\}.$$

The last term

$$=(-)^m\frac{(2m!)}{(m!)^2}\cdot\frac{1}{2^{2m}}=(-)^m\frac{(m-\frac{1}{2})(m-\frac{3}{2})\ldots\frac{1}{2}}{m!},$$

and the expression in brackets is equal to the coefficient of t^m in

$$(1+t)^{p-1}-(m-\tfrac{1}{2})t(1+t)^{p-1}+\frac{(m-\frac{1}{2})(m-\frac{3}{2})}{2!}t^2(1+t)^{p-1}\ldots+(-)^m\frac{(m-\frac{1}{2})\ldots\frac{1}{2}}{m!}t^m(1+t)^{p-m-1}$$

$$=(1+t)^{p-1}\left\{1-\frac{t}{1+t}\right\}^{m-1}\qquad=(1+t)^{p-m}.$$

The coefficient of t^m in the expansion of $(1+t)^{p-m}$ is equal to

$$\frac{(p-m)(p-m-1)\ldots(p-2m+1)}{m!},$$

and therefore the coefficient of $a^{2m}x^{2m}$ in $x^p.\mathrm{U}e^{-ax}$ is equal to

$$\frac{1}{(2m)!}\frac{(p-m)(p-m-1)\ldots(p-2m+1)}{(p-\frac{1}{2})(p-\frac{3}{2})\ldots(p-m+\frac{1}{2})},$$

which is the coefficient of $a^{2m}x^{2m}$ in $x^p.\mathrm{P}$ when the factors $p(p-1)(p-2)\ldots(p-m+1)$ are divided out from the numerator and denominator.

Similarly, if $n=2m+1$, the coefficient of $a^{2m+1}x^{2m+1}$ in $x^p.\mathrm{U}e^{-ax}$ is found to be equal to

$$-\frac{1}{(2m+1)!}\frac{(p-m-1)(p-m-2)\ldots(p-2m)}{(p-\frac{1}{2})(p-\frac{3}{2})\ldots(p-m+\frac{1}{2})},$$

which is the coefficient of $a^{2m+1}x^{2m+1}$ in $x^p.\mathrm{P}$ when the factors $p(p-1)\ldots(p-m)$ are divided out from the numerator and denominator.

Thus, if $p=i$, the coefficients of the terms involving x^{i+1}, x^{i+2}, $\ldots$ x^{2i} in the series in P vanish, and we have $\mathrm{U}=\mathrm{P}'+g\mathrm{Q}$.

(3°) If p = a negative integer $=-i-1$, then Q and S involve zero terms, and, denoting by Q′ and S′ the values of Q and S when the series are supposed to terminate at the term preceding the first term involving a zero factor in the numerator, V, Q, and S become equal to U, P, and R when p is put equal to i, that is, to the U, P, and R of (2°) and *vice versâ*. In this case, therefore,

$$\mathrm{Q}=\mathrm{Q}'+g\mathrm{P}=\mathrm{V}=\mathrm{S}=\mathrm{S}'-g\mathrm{R},$$

and

$$\mathrm{P}=\mathrm{R}=\mathrm{U}.$$

The relations between the particular integrals in the three cases are therefore

(1°) p not = an integer,

$$\mathrm{P}=\mathrm{R}=\mathrm{U},\qquad\mathrm{Q}=\mathrm{S}=\mathrm{V}.$$

(2°) p = a positive integer,

$$P=R=U=\tfrac{1}{2}(P'+R'), \qquad Q=S=V=\frac{1}{2g}(R'-P');$$

(3°) p = a negative integer,

$$P=R=U=\frac{1}{2g}(S'-Q'), \qquad Q=S=V=\tfrac{1}{2}(Q'+S');$$

6. If we suppose the series always to terminate directly a zero factor appears in a numerator (so that P′, Q′, R′, S′ are now denoted by P, Q, R, S), the relations are

(1°) p not = an integer,

$$P=R=U, \qquad Q=S=V;$$

(2°) p = a positive integer,

$$Q=S=V=\frac{1}{2g}(R-P), \qquad U=\tfrac{1}{2}(P+R);$$

(3°) p = a negative integer,

$$P=R=U=\frac{1}{2g}(S-Q), \qquad V=\tfrac{1}{2}(Q+S);$$

The change of form of the relations, which in this mode of statement appears so remarkable, does not, as we have seen, occur if the series be supposed to extend to infinity in all cases.

It may be observed that it is clear from the manner in which the series were obtained in arts. 1 and 2 that we are always at liberty to stop at the term immediately preceding the first term containing a zero factor in the numerator, as this finite portion of the series satisfies the differential equation, and that the second series obtained by allowing the terms to recommence and to extend to infinity also satisfies the differential equation.

The phrase "term preceding the first term containing a zero factor in the numerator" has been used in preference to "term preceding the first zero term" in order to include the cases of $p=0$ or $p=-1$, in which no zero term occurs.

7. It was shown in art. 5 that

$$\left(1-\frac{p}{p}ax+\frac{p(p-1)}{p(p-\frac{1}{2})}\frac{a^2x^2}{2!}-\frac{p(p-1)(p-2)}{p(p-\frac{1}{2})(p-1)}\frac{a^3x^3}{3!}+\&c.\right)e^{ax}$$

$$=1-\frac{1}{p-\frac{1}{2}}\frac{a^2x^2}{2^2}+\frac{1}{(p-\frac{1}{2})(p-\frac{3}{2})}\frac{a^4x^4}{2^4.2!}-\frac{1}{(p-\frac{1}{2})(p-\frac{3}{2})(p-\frac{5}{2})}\frac{a^6x^6}{2^6.3!}+\&c.$$

Putting $2p=-m-1$ in this identity, we have

$$\left(1-\frac{m+1}{m+1}ax+\frac{(m+1)(m+3)}{(m+1)(m+2)}\frac{a^2x^2}{2!}-\frac{(m+1)(m+3)(m+5)}{(m+1)(m+2)(m+3)}\frac{a^3x^3}{3!}+\&c.\right)e^{ax}$$
$$=1+\frac{1}{m+2}\frac{a^2x^2}{2}+\frac{1}{(m+2)(m+4)}\frac{a^4x^4}{2^2.2!}+\frac{1}{(m+2)(m+4)(m+6)}\frac{a^6x^6}{2^3.3!}+\&c.$$

The right-hand side of this equation is unaltered by a change of sign of x, and therefore, putting $a=1$,

$$\left(1-\frac{m+1}{m+1}x+\frac{(m+1)(m+3)}{(m+1)(m+2)}\frac{x^2}{2!}-\frac{(m+1)(m+3)(m+5)}{(m+1)(m+2)(m+3)}\frac{x^3}{3!}+\&c.\right)e^x$$
$$=\left(1+\frac{m+1}{m+1}x+\frac{(m+1)(m+3)}{(m+1)(m+2)}\frac{x^2}{2!}+\frac{(m+1)(m+3)(m+5)}{(m+1)(m+2)(m+3)}\frac{x^3}{3!}+\&c.\right)e^{-x},$$

which is true for all values of m, except m = a negative even integer.

Writing n in place of $m+1$, it follows that

$$e^{2x}=\frac{1+x+\frac{n+2}{n+1}\frac{x^2}{2!}+\frac{(n+2)(n+4)}{(n+1)(n+2)}\frac{x^3}{3!}+\&c.}{1-x+\frac{n+2}{n+1}\frac{x^2}{2!}-\frac{(n+2)(n+4)}{(n+1)(n+2)}\frac{x^3}{3}+\&c.},$$

which is true for all values of n, except n = a negative uneven integer. Several deductions from this formula are given in a paper "Generalised Form of Certain Series" ('Proceedings of the London Mathematical Society,' vol. ix., pp. 197–204, 1878).

§ II.

Integration of the differential equation when p = an integer. Arts. 8–15.

8. A particular integral of the equation

$$\frac{d^2u}{dx^2}-a^2u=\frac{h^2}{x^3}\frac{d^2u}{dh^2}$$

is

$$u=e^{a\sqrt{(x^2+xh)}},$$

for, from this value of u we find at once by differentiation

$$\frac{d^2u}{dx^2}=a^2u\frac{(x+\frac{1}{2}h)^2}{x^2+xh}-au\frac{\frac{1}{4}h^2}{(x^2+xh)^{\frac{3}{2}}},$$

$$\frac{d^2u}{dh^2}=a^2u\frac{\frac{1}{4}x^2}{x^2+xh}-au\frac{\frac{1}{4}x^2}{(x^2+xh)^{\frac{3}{2}}},$$

whence

$$\frac{d^2u}{dx^2}-a^2u=\frac{h^2}{x^2}\frac{d^2u}{dh^2}.$$

9. Let the above value of u be expanded in powers of h, so that

$$u=e^{a\sqrt{(x^2+xh)}}=P_0+P_1h+P_2h^2\ldots+P_ih^i+P_{i+1}h^{i+1}+\&c.,$$

then

$$\frac{d^2u}{dx^2}-a^2u=\ldots+\left(\frac{d^2P_{i+1}}{dx^2}-a^2P_{i+1}\right)h^{i+1}+\&c.,$$

$$\frac{h^2}{x^2}\frac{d^2u}{dh^2}=\ldots+\frac{(i+1)i}{x^2}h^{i+1}+\&c.,$$

and therefore P_{i+1} satisfies the differential equation

$$\frac{d^2u}{dx^2}-a^2u=\frac{i(i+1)}{x^2}u.$$

Thus the general integral of this differential equation is

$$u=\text{A. coefficient of } h^{i+1} \text{ in expansion of } e^{a\sqrt{(x^2+xh)}}$$
$$+\text{B. coefficient of } h^{i+1} \text{ in expansion of } e^{-a\sqrt{(x^2+xh)}}$$

The particular integrals to which the different modes of expansion of $e^{a\sqrt{(x^2+xh)}}$ lead will now be examined, and connected with the forms already obtained in § I.

10. The coefficient of h^{i+1} in the expansion of $e^{a\sqrt{(x^2+xh)}}$ is equal to the coefficient of h^{i+1} in

$$1+a(x^2+xh)^{\frac{1}{2}}+\frac{a^2}{2!}(x^2+xh)+\frac{a^3}{3!}(x^2+xh)^{\frac{1}{2}}+\frac{a^4}{4!}(x^2+xh)^2+\frac{a^5}{5!}(x^2+xh)^{\frac{1}{2}}+\&c.,$$

and the coefficient of h^{i+1} in $(x^2+xh)^{\frac{1}{2}(2n-1)}$

$$=\frac{(n-\frac{1}{2})(n-\frac{3}{2})\ldots(n-i-\frac{1}{2})}{(i+1)!}x^{2n-i-2}.$$

Thus the coefficient of h^{i+1} in the terms involving uneven powers of a

$$=\tfrac{1}{2}\frac{(-1)^i}{(i+1)!}\tfrac{1}{2}.\tfrac{3}{2}\ldots(i-\tfrac{1}{2})x^{-i}\left\{a-\frac{a^3}{3!}\frac{\frac{3}{2}}{i-\frac{1}{2}}x^3+\frac{a^5}{5!}\frac{\frac{3}{2}.\frac{5}{2}}{(i-\frac{1}{2})(i-\frac{3}{2})}x^4-\&c.\right\}$$

$$=\tfrac{1}{2}a\frac{(-1)^i}{(i+1)!}\tfrac{1}{2}.\tfrac{3}{2}\ldots(i-\tfrac{1}{2})x^{-i}\left\{1-\frac{1}{i-\frac{1}{2}}\frac{a^2x^2}{2^2}+\frac{1}{(i-\frac{1}{2})(1-\frac{3}{2})}\frac{a^4x^4}{2^4 2!}-\&c.\right\}=\lambda U,$$

where

$$\lambda=\tfrac{1}{2}a\frac{(-1)^i}{(i+1)!}\tfrac{1}{2}.\tfrac{3}{2}\dots(i-\tfrac{1}{2})=(-)^i\frac{1.3.5\dots(2i-1)}{2.4.6\dots(2i+2)}a.$$

Of the terms involving even powers of a the first that contains a term in h^{i+1} is

$$\frac{a^{2i+2}}{(2i+2)!}x^{i+1}(x+h)^{i+1},$$

so that the coefficient of h^{i+1} in the terms involving even powers of a

$$=\frac{a^{2i+2}}{(2i+2)!}x^{i+1}+\frac{a^{2i+4}}{(2i+4)!}(i+2)x^{i+3}+\frac{a^{2i+6}}{(2i+6)!}\frac{(i+2)(i+3)}{2!}x^{i+5}+\&c.$$

$$=\frac{a^{2i+2}}{(2i+2)!}x^{i+1}\left\{1+\frac{i}{i+\frac{3}{2}}\frac{a^2x^2}{2!}+\frac{1}{(i+\frac{3}{2})(i+\frac{5}{2})}\frac{a^4x^4}{2^2 2!}+\&c.\right\}=\frac{a^{2i+2}}{(2i+2)!}\mathrm{V}.$$

The complete coefficient of h^{i+1} in the expansion of $e^{a\sqrt{(x^2+xh)}}$ therefore

$$=\lambda\mathrm{U}+\frac{a^{2i+2}}{(2i+2)!}\mathrm{V}=\lambda\left\{\mathrm{U}+(-)^i a^{2i+1}\frac{2i+1}{(1.3.5\dots 2i+1)^2}\mathrm{V}\right\}$$

$$=\lambda\{\mathrm{U}-g\mathrm{V}\},$$

g being the same as in art 5.

11. Now

$$e^{a\sqrt{(x^2+xh)}}=e^{ax}.\ e^{a\{\sqrt{(x^2+xh)}-x\}}=e^{-ax}.\ e^{a\{\sqrt{(x^2+xh)}+x\}},$$

and we obtain other forms of the integral by finding the coefficients of h^{i+1} in the expansion of $e^{a\{\sqrt{(x^2+xh)}-x\}}$ and of $e^{a\{\sqrt{(x^2+xh)}+x\}}$, and multiplying them by e^{ax} and e^{-ax} respectively.

It is well known that

$$\left\{1-\sqrt{(1-4t)}\right\}^n=2^n t^n\left\{1+nt+\frac{n(n+3)}{2!}t^2+\frac{n(n+4)(n+5)}{3!}t^3+\&c.\right\},$$

and

$$\left\{1+\sqrt{(1-4t)}\right\}^n=2^n\left\{1-nt+\frac{n(n-3)}{2!}t^2-\frac{n(n-4)(n-5)}{3!}t^3+\&c.\right\},$$

where in the second series, if n is an even positive integer the coefficients of the $\frac{1}{2}n-1$ terms involving $t^{\frac{1}{2}n+1}$, $t^{\frac{1}{2}n+2}\dots t^{n-1}$ are zero, and if n is an uneven positive integer the coefficients of the $\frac{1}{2}(n-1)$ terms involving $t^{\frac{1}{2}(n+1)}$, $t^{\frac{1}{2}(n+3)}\dots t^{n-1}$ are zero.

Putting $t=-\dfrac{h}{4x}$, these formulæ become

$$\{\sqrt{(x^2+xh)}-x\}^n=\frac{1}{2^n}h^n\left\{1-n\frac{h}{4x}+\frac{n(n+3)}{2!}\frac{h^2}{4^2x^2}-\frac{n(n+4)(n+5)}{3!}\frac{h^3}{4^3x^3}+\&c.\right\},$$

$$\{\sqrt{(x^2+xh)}+x\}^n=2^n x^n\left\{1+n\frac{h}{4x}+\frac{n(n-3)}{2!}\frac{h^2}{4^2x^2}+\frac{n(n-4)(n-5)}{3!}\frac{h^3}{4^3x^3}+\&c.\right\}.$$

The coefficient of h^{i+1} in $\{\sqrt{(x^2+xh)}-x\}^n$ therefore

$$=\frac{1}{2^n}(-)^{i+1-n}\frac{n(i+2)(i+3)\ldots(2i+1-n)}{(i+1-n)!}\frac{1}{4^{i+1-n}x^{i+1-n}}=\frac{(-1)^{i+1}}{4^{i+1}x^{i+1}}\cdot(-)^n\frac{(i+2)\ldots(2i+1-n)}{(i+1-n)!}n2^nx^n,$$

and the coefficient of h^{i+1} in $\{\sqrt{(x^2+xh)}+x\}^n$

$$=2^nx^n\frac{n(n-i-2)(n-i-3)\ldots(n-2i-1)}{(i+1)!}\frac{1}{4^{i+1}x^{i+1}}=\frac{(-1)^i}{4^{i+1}x^{i+1}}\frac{(i+2-n)\ldots(2i+1-n)}{(i+1)!}n2^nx^n.$$

12. The coefficient of h^{i+1} in $e^{\alpha\{\sqrt{(x^2+xh)}-x\}}$, that is in

$$1+\alpha\{\sqrt{(x^2+xh)}-x\}+\frac{\alpha^2}{2!}\{\sqrt{(x^2+xh)}-x\}^2+\frac{\alpha^3}{3!}\{\sqrt{(x^2+xh)}-x\}^3+\&c.,$$

is, by the last article, equal to

$$\frac{(-1)^i}{4^{i+1}x^{i+1}}\Big\{a\frac{(i+2)\ldots 2i}{i!}2x-\frac{a^2}{2!}\frac{(i+2)\ldots(2i-1)}{(i-1)!}2.2^2x^2 +\frac{a^3}{3!}\frac{(i+2)\ldots(2i-3)}{(i-2)!}3.2^3x^3-\ldots+(-)^i\frac{a^{i+1}}{(i+1)!}2^{i+1}x^{i+1}\Big\},$$

for, when n is greater than $i+1$, there is no term involving h^{i+1}.

This expression

$$=(-)^i\tfrac{1}{2}a\frac{(i+2)\ldots 2i}{4^i.i!}\frac{1}{x^i}\Big\{1-a\frac{i}{2i}2x+\frac{a^2}{2!}\frac{i(i-1)}{2i(2i-1)}2^2x^2\ldots+(-)^i\frac{a^i}{i!}\frac{i!}{2i(2i-1)\ldots(i+1)}2^ix^i\Big\},$$

$$=\lambda\frac{1}{x^i}\Big\{1-\frac{i}{i}ax+\frac{i(i-1)}{i(i-\frac{1}{2})}\frac{a^2x^2}{2!}\ldots+(-)^i\frac{i(i-1)\ldots\{i-(i-1)\}}{i(i-\frac{1}{2})\ldots\{i-\frac{1}{2}(i-1)\}}\frac{a^ix^i}{i!}\Big\},$$

for the constant multiplier

$$=(-)^i\tfrac{1}{2}a\frac{(i+2)\ldots 2i}{4^i.i!}=(-)^i\frac{(2i)!}{(2.4.6\ldots 2i)^2(2i+2)}a=(-)^i\frac{1.3.5\ldots(2i-1)}{2.4.6\ldots(2i+2)}a,$$

which is the quantity denoted by λ in art. 10.

The coefficient of h^{i+1} in the expansion of $e^{\alpha x}.e^{\alpha\{\sqrt{(x^2+xh)}-x\}}$ is therefore $=\lambda P'$.

13. The coefficient of h^{i+1} in $e^{\alpha\{\sqrt{(x^2+xh)}+x\}}$, that is in

$$1+\alpha\{\sqrt{(x^2+xh)}+x\}+\frac{\alpha^2}{2!}\{\sqrt{(x^2+xh)}+x\}^2+\frac{\alpha^3}{3!}\{\sqrt{(x^2+xh)}+x\}^3+\&c.,$$

is, by art. 10, equal to

$$\frac{(-1)^i}{(i+1)!}\frac{1}{4^{i+1}x^{i+1}}\Big[a\{(i+1)\dots 2i\}2x+\frac{a^2}{2!}\{i\dots(2i-1)\}2.2^2x^2\dots$$

$$+\frac{a^{i+1}}{(i+1)!}\{1.2\dots i\}(i+1)2^{i+1}x^{i+1}$$

$$+\frac{a^{2i+2}}{(2i+2)!}\{(-i)(-i+1)\dots(-1)\}(2i+2)2^{2i+2}x^{2i+2}$$

$$+\frac{a^{2i+3}}{(2i+3)!}\{(-i-1)\dots(-2)\}(2i+3)2^{2i+3}x^{2i+3}+\&c.\Big]$$

$$=\lambda\frac{1}{x^i}\left\{1+\frac{i}{i}ax+\frac{i(i-1)}{i(i-\frac{1}{2})}\frac{a^2x^2}{2!}\dots+\frac{i(i-1)\dots\{i-(i-1)\}}{i(i-\frac{1}{2})\dots\{i-\frac{1}{2}(i-1)\}}\frac{a^ix^i}{i!}\right\}$$

$$+(-)^i\frac{i!}{(2i+1)!}a^{2i+2}x^{i+1}\left\{1+\frac{i+1}{i+1}ax+\frac{(i+1)(i+2)}{(i+1)(i+\frac{3}{2})}\frac{a^2x^2}{2!}+\&c.\right\}.$$

The coefficient of h^{i+1} in the expansion of $e^{-ax}.e^{a\{\sqrt{(x^2+xh)}+x\}}$ is therefore

$$\lambda R'+\frac{a^{2i+2}}{(i+1).(2i+1)!}S;$$

and we have

$$\frac{1}{\lambda}\frac{a^{2i+2}}{(i+1).(2i+1)!}=(-)^i\frac{2(2i+1)}{\{1.3.5\dots(2i+1)\}^2}a^{2i+1}=-2g,$$

so that the coefficient of h^{i+1}

$$=\lambda(R'-2gS).$$

14. Thus the three forms of the same integral which are obtained by the expansion of

$$e^{a\sqrt{(x^2+xh)}},\qquad e^{ax}.e^{a\{\sqrt{(x^2+xh)}-x\}},\qquad e^{-ax}.e^{a\{\sqrt{(x^2+xh)}+x\}}$$

are

$$U-gV,\qquad P',\qquad R'-2gS.$$

Changing the sign of a, we obtain as the coefficient of h^{i+1} in the expansion of

$$e^{-a\sqrt{(x^2+xh)}},\qquad e^{-ax}.e^{-a\{\sqrt{(x^2+xh)}-x\}},\qquad e^{ax}.e^{-a\{\sqrt{(x^2+xh)}+x\}}$$

the values $-\lambda(U+gV)$, $-\lambda R'$, $-\lambda(P'+2gQ)$, giving the three equal integrals

$$U+gV,\qquad R',\qquad P'+2gQ.$$

Therefore

$$U-gV=P'=R'-2gS,$$

$$U+gV=R'=P'+2gQ.$$

whence

$$U=\tfrac{1}{2}(P'+R'), \qquad Q=S=V=\frac{1}{2g}(R'-P'),$$

which agree with the relations found for the case of $p=$ a positive integer in art. 5.

If p is a negative integer $=-i-1$, then $p(p+1)=i(i+1)$; we may therefore replace i by $-i-1$ throughout in the integrals just obtained, and thus deduce the system of integrals considered in (3°) of art. 5.

15. It may be observed that, since the series for $\{1-\sqrt{(1-4t)}\}^n$ and $\{1+\sqrt{(1-4t)}\}^n$ in art. 11 terminate and recommence when n is respectively a negative or positive integer, it is evident that the solutions in series of the differential equation satisfied by them will present points of similarity to the solutions Q and P of (1). The former differential equation is

$$t(1-4t)\frac{d^2u}{dt^2}+\{(4n-6)t-n+1\}\frac{du}{dt}-n(n-1)u=0,$$

and its integration in series is considered in a paper "Example Illustrative of a Point in the Solution of Differential Equations in Series" ('Messenger of Mathematics,' vol. viii., pp. 20–23).

§ III.

Transformations of the original differential equation. RICCATI'S *equation.* Arts. 16, 17.

16. If the differential equation

$$\frac{d^2u}{dx^2}-a^2u=\frac{p(p+1)}{x^2}u \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

is transformed by assuming $u=x^{-p}v$, it becomes

$$\frac{d^2v}{dx^2}-\frac{2p}{x}\frac{dv}{dx}-a^2v=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2).$$

This equation therefore admits of integration in a finite form when $p=$ an integer, and the six particular integrals U_1, V_1, P_1, Q_1, R_1, S_1, which are equal respectively to x^pU, x^pV, x^pP, x^pQ, x^pR, x^pS are connected with one another, in the different cases, by the same relations as those found for U, V, P, Q, R, S in art. 5.

If we put $2p=n-1$, so that the differential equation becomes

$$\frac{d^2v}{dx^2}-\frac{n-1}{x}\frac{dv}{dx}-a^2v=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3),$$

then the six integrals take the forms

$$U_1= \quad 1-\frac{1}{n-2}\frac{a^2x^2}{2}+\frac{1}{(n-2)(n-4)}\frac{a^4x^4}{2^2.2!}-\frac{1}{(n-2)(n-4)(n-6)}\frac{a^6x^6}{2^3.3!}+\&c.,$$

$$V_1=x^n\left\{1+\frac{1}{n+2}\frac{a^2x^2}{2}+\frac{1}{(n+2)(n+4)}\frac{a^4x^4}{2^2.2!}+\frac{1}{(n+2)(n+4)(n+6)}\frac{a^6x^6}{2^3.3!}+\&c.\right\},$$

$$P_1= \quad \left\{1-\frac{n-1}{n-1}ax+\frac{(n-1)(n-3)}{(n-1)(n-2)}\frac{a^2x^2}{2!}-\frac{(n-1)(n-3)(n-5)}{(n-1)(n-2)(n-3)}\frac{a^3x^3}{3!}+\&c.\right\}e^{ax},$$

$$Q_1=x^n\left\{1-\frac{n+1}{n+1}ax+\frac{(n+1)(n+3)}{(n+1)(n+2)}\frac{a^2x^2}{2!}\quad\frac{(n+1)(n+3)(n+5)}{(n+1)(n+2)(n+3)}\frac{a^3x^3}{3!}+\&c.\right\}e^{ax},$$

$$R_1= \quad \left\{1+\frac{n-1}{n-1}ax+\frac{(n-1)(n-3)}{(n-1)(n-2)}\frac{a^2x^2}{2!}+\frac{(n-1)(n-3)(n-5)}{(n-1)(n-2)(n-3)}\frac{a^3x^3}{3!}+\&c.\right\}e^{-ax},$$

$$S_1=x^n\left\{1+\frac{n+1}{n+1}ax+\frac{(n+1)(n+3)}{(n+1)(n+2)}\frac{a^2x^2}{2!}+\frac{(n+1)(n+3)(n+5)}{(n+1)(n+2)(n+3)}\frac{a^3x^3}{3!}+\&c.\right\}e^{-ax}.$$

The differential equation admits of integration in a finite form if $n=$ an uneven integer, and the relations between the particular integrals are the same as in art. 5, viz., accented letters denoting the terminated series as before,

(1°.) n not $=$ an integer,

$$P_1=R_1=U_1, \qquad Q_1=S_1=V_1;$$

(2°.) $n=$ a positive uneven integer,

$$P_1=R_1=U_1=\tfrac{1}{2}(P_1'+R_1'), \qquad Q_1=S_1=V_1=\frac{1}{2g_1}(R_1'-P_1');$$

(3°.) $n=$ a negative uneven integer,

$$P_1=R_1=U_1=\frac{1}{2g_1}(S_1'-Q_1'), \qquad Q_1=S_1=V_1=\tfrac{1}{2}(Q_1'+S_1');$$

where

$$g_1=(-)^{\frac{1}{2}(n+1)}\frac{n}{1^2.3^2.5^2\ldots n^2}a^n.$$

This is perhaps the simplest form in which the six integrals can be exhibited; and, having regard merely to the simplicity of the series and to the expression of the manner in which they are related to one another, (3) should be preferred as the standard form of the differential equation both to the original form (1) and to RICCATI's equation (4), which is considered in the next article. The form (3) is that adopted by BACZ in his memoir (cf. § VIII.).

It may be observed that if $p=i$, a positive integer, the differential equation (2) is satisfied by $v=\frac{1}{x}\times$ coefficient of h^{i+1} in the expansion of $e^{ax\sqrt{(1+h)}}$, and if $p=-i$, by $v=x\times$ coefficient of h^{i+1} in the expansion of $e^{a\sqrt{(x^2+h)}}$; these results follow from § II.

17. Transforming the equation (3) by assuming $x=nz^{\frac{1}{n}}$, it becomes

$$z^{2-\frac{2}{n}}\frac{d^2v}{dz^2}-a^2v=0,$$

or, putting $n=\frac{1}{q}$,

$$\frac{d^2v}{dz^2}-a^2z^{2q-2}v=0 \qquad \ldots\ldots\ldots\ldots \quad (4).$$

RICCATI's equation in its original form is

$$\frac{dy}{dz}+by^2=cz^m;$$

it may without loss of generality be written

$$\frac{dy}{dz}+y^2=z^m,$$

and, putting $y=\frac{1}{v}\frac{dv}{dz}$, it becomes

$$\frac{d^2v}{dz^2}-z^mv=0.$$

Thus (4) is the equation derived from

$$\frac{dy}{dz}+y^2=a^2z^{2q-2}$$

by assuming $y=\frac{1}{v}\frac{dv}{dz}$, and it is convenient to regard it as the standard form of RICCATI's equation.

The six particular integrals of (4) are

$$U_2 = 1+\frac{a^2z^{2q}}{2q(2q-1)}+\frac{a^4z^{4q}}{2q(2q-1)4q(4q-1)}+\frac{a^6z^{6q}}{2q(2q-1)4q(4q-1)6q(6q-1)}+\&c.,$$

$$V_2 = z\left\{1+\frac{a^2z^{2q}}{2q(2q+1)}+\frac{a^4z^{4q}}{2q(2q+1)4q(4q+1)}+\frac{a^6z^{6q}}{2q(2q+1)4q(4q+1)6q(6q+1)}+\&c.\right\},$$

$$P_2 = \left\{1-\frac{q-1}{q(q-1)}az^q+\frac{(q-1)(3q-1)}{q(q-1)2q(2q-1)}a^2z^{2q}-\frac{(q-1)(3q-1)(5q-1)}{q(q-1)2q(2q-1)3q(3q-1)}a^3z^{3q}+\&c.\right\}e^{\frac{a}{q}z^q},$$

$$Q_2 = z\left\{1-\frac{q+1}{q(q+1)}az^q+\frac{(q+1)(3q+1)}{q(q+1)2q(2q+1)}a^2z^{2q}-\frac{(q+1)(3q+1)(5q+1)}{q(q+1)2q(2q+1)3q(3q+1)}a^3z^{3q}+\&c.\right\}e^{\frac{a}{q}z^q},$$

$$R_2 = \left\{1+\frac{q-1}{q(q-1)}az^q+\frac{(q-1)(3q-1)}{q(q-1)2q(2q-1)}a^2z^{2q}+\frac{(q-1)(3q-1)(5q-1)}{q(q-1)2q(2q-1)3q(3q-1)}a^3z^{3q}+\&c.\right\}e^{-\frac{a}{q}z^q},$$

$$S_2 = z\left\{1+\frac{q+1}{q(q+1)}az^q+\frac{(q+1)(3q+1)}{q(q+1)2q(2q+1)}a^2z^{2q}+\frac{(q+1)(3q+1)(5q+1)}{q(q+1)2q(2q+1)3q(3q+1)}a^3z^{3q}+\&c.\right\}e^{-\frac{a}{q}z^q}.$$

The differential equation admits of integration in a finite form if q = the reciprocal of an uneven integer, and, the terminated series being denoted by accented letters as before, the relations between the particular integrals are the same as in art. 5, viz.

(1°) q not = the reciprocal of an uneven integer,

$$P_2=R_2=U_2, \quad Q_2=S_2=V_2;$$

(2°) q = the reciprocal of an uneven positive integer,

$$P_2=R_2=U_2=\tfrac{1}{2}(P_2'+R_2'), \quad Q_2=S_2=V_2=\left(\frac{1}{q}\right)^{1-\frac{1}{q}}\frac{1}{2g_2}(R_2'-P_2');$$

(3°) q = the reciprocal of an uneven negative integer,

$$P_2=R_2=U_2=\left(\frac{1}{q}\right)^{1+\frac{1}{q}}\frac{1}{2g_2}(S_2'-Q_2'), \quad Q_2=S_2=V_2=\tfrac{1}{2}(Q_2'+S_2');$$

where

$$g_2=\frac{(-1)^{\frac{1}{2}\left(1+\frac{1}{q}\right)}}{1^2.3^2.5^2\ldots\frac{1}{q^2}}a^{\frac{1}{q}}.$$

The integrals P_2, Q_2, R_2, S_2 were given by CAYLEY in the 'Philosophical Magazine,' Fourth series, vol. 36, pp. 348–351 (November, 1868).

§ IV.

Special forms of the particular integrals in the cases in which the differential equations admit of integration in a finite form. Arts. 18, 19.

18. When the differential equations admit of integration in series containing a finite number of terms, these finite particular integrals may be presented in another form by commencing the terminating series at the other end.

Thus in the case of the differential equation

$$\frac{d^2u}{dx^2}-a^2u=\frac{p(p+1)}{x^2}u,$$

if p is a positive integer, the particular integral

$$\frac{1}{x^p}\left\{1-\frac{p}{p}ax+\frac{p(p-1)}{p(p-\frac{1}{2})}\frac{a^2x^2}{2!}\cdots+(-)^{p-1}\frac{p(p-1)\ldots2}{p(p-\frac{1}{2})\ldots\frac{1}{2}(p+2)}\frac{a^{p-1}x^{p-1}}{(p-1)!}\right.$$
$$\left.+(-)^p\frac{p(p-1)\ldots1}{p(p-\frac{1}{2})\ldots\frac{1}{2}(p+1)}\frac{a^px^p}{p!}\right\}e^{ax}$$

$$=(-)^p\frac{2^pa^p}{(p+1)\ldots2p}\left\{1-\frac{p}{1}\cdot\frac{1}{2}(p+1)\frac{1}{ax}+\frac{p(p-1)}{1.2}\frac{1}{2}(p+1)\frac{1}{2}(p+2)\frac{1}{a^2x^2}\cdots\right.$$
$$\left.+(-)^p\frac{1}{2}(p+1)\frac{1}{2}(p+2)\ldots\frac{1}{2}(2p)\frac{1}{a^px^p}\right\}e^{ax}$$

$$=(-)^p\frac{2^pa^p}{(p+1)\ldots2p}\left\{1-\frac{p(p+1)}{2}\frac{1}{ax}+\frac{(p-1)p(p+1)(p+2)}{2.4}\frac{1}{a^2x^2}\cdots\right.$$
$$\left.+(-)^p\frac{1.2\ldots2p}{2.4\ldots2p}\frac{1}{a^px^p}\right\}e^{ax};$$

so that, if $p=$ an integer, the finite particular integrals are

$$\left\{1-\frac{p(p+1)}{2}\frac{1}{ax}+\frac{(p-1)p(p+1)(p+2)}{2.4}\frac{1}{a^2x^2}-\&c.\right\}e^{ax}$$

(the series being continued till it terminates of itself through the terms all containing a zero factor), and a similar expression derived from this by changing the sign of a.

19. Similarly, if $n=$ an uneven integer, then

$$x^{\frac{1}{2}(n-1)}\left\{1-(n^2-1^2)\left(\frac{1}{8ax}\right)+\frac{(n^2-1^2)(n^2-3^2)}{1.2}\left(\frac{1}{8ax}\right)^2-\frac{(n^2-1^2)(n^2-3^2)(n^2-5^2)}{1.2.3}\left(\frac{1}{8ax}\right)^3+\&c.\right\}e^{ax}$$

and a similar expression derived from this by changing the sign of a, are particular integrals of the differential equation

$$\frac{d^2v}{dx^2}-\frac{n-1}{x}\frac{dv}{dx}-a^2v=0.$$

In the case of RICCATI's equation,

$$\frac{d^2v}{dz^2}-a^2z^{2q-2}v=0,$$

if $q=$ the reciprocal of an uneven integer, the two particular integrals are

$$z^{\frac{1}{2}(1-q)}\left\{1+\frac{q^2-1}{q}\left(\frac{1}{8az^q}\right)+\frac{(q^2-1)(3^2q^2-1)}{q.2q}\left(\frac{1}{8az^q}\right)^2+\frac{(q^2-1)(3^2q^2-1)(5^2q^2-1)}{q.2q.3q}\left(\frac{1}{8az^q}\right)^3+\&c.\right\}e^{\pm\frac{2}{q}az^q}$$

and a similar expression derived from this by changing the sign of a.

These appear to be the best forms in which the integrals can be presented when the equations admit of solution in a finite form: but they do not suggest the solutions for the general cases when the letters are unrestricted. The series ultimately become divergent when they do not terminate.

§ V.

Evaluation of definite integrals satisfying the differential equations. Arts. 20–28.

20. It was shown by POISSON* that the definite integral

$$y=\int_0^\infty e^{-x^m-\frac{bz^m}{x^m}}dx \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

satisfies the RICCATI's equation

$$\frac{d^2y}{dz^2}=m^2bz^{m-2}y \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6),$$

so that the value of the integral must be of the form AU_2+BV_2, where U_2, V_2 are the same as in art. 17, and $a^2=m^2b$, $q=\frac{1}{2}m$; it remains to determine the constants A and B.

* 'Journal de l'École Polytechnique,' Cahier xvi. (vol. ix., 1813), p. 237.

It is however more convenient in the first place to consider the definite integral in the form

$$y=\int_0^\infty x^{n-1}e^{-x^2-\frac{\alpha^2}{x^2}}\,dx,$$

which is obtained by transforming (5) by the substitution $x^m=x'^2$, for the integral (5) thus becomes

$$\int_0^\infty \frac{2}{m}x^{\frac{2}{m}n-1}e^{-x^2-\frac{b\alpha^m}{x^2}}\,dx.$$

Comparing (6) with the standard form (4) of RICCATI'S equation in art. 17, we have $m=2q$, so that $\frac{2}{m}=\frac{1}{q}=n$, and $m^2b=a^2$.

Let $bx^m=\alpha^2$; then $z=b^{-\frac{1}{2}n}\alpha^n$, and we see that the definite integral

$$y=\int_0^\infty x^{n-1}e^{-x^2-\frac{\alpha^2}{x^2}}\,dx$$

satisfies the differential equation

$$\frac{d^2y}{d\alpha^2}-\frac{n-1}{\alpha}\frac{dy}{d\alpha}-4y=0.$$

The value of this definite integral is therefore of the form AU_1+BV_1, where U_1, V_1 are the same as in art. 16, α being substituted for x and a put $=2$: viz., writing

$$M=1-\frac{1}{n-2}(2\alpha^2)+\frac{1}{(n-2)(n-4)}\frac{(2\alpha^2)^2}{2!}-\frac{1}{(n-2)(n-4)(n-6)}\frac{(2\alpha^2)^3}{3!}+\&c.,$$

$$N=1+\frac{1}{n+2}(2\alpha^2)+\frac{1}{(n+2)(n+4)}\frac{(2\alpha^2)^2}{2!}+\frac{1}{(n+2)(n+4)(n+6)}\frac{(2\alpha^2)^3}{3!}+\&c.,$$

then

$$\int_0^\infty x^{n-1}e^{-x^2-\frac{\alpha^2}{x^2}}\,dx=AM+B\alpha^nN.$$

Suppose n positive, and put $\alpha=0$; we thus find

$$A=\int_0^\infty x^{n-1}e^{-x^2}dx=\tfrac{1}{2}\Gamma(\tfrac{1}{2}n),$$

and therefore

$$\int_0^\infty x^{n-1}e^{-x^2-\frac{\alpha^2}{x^2}}\,dx=\tfrac{1}{2}\Gamma(\tfrac{1}{2}n)M+\alpha^n\phi(n)N.$$

Transform the integral by assuming $x=\frac{\alpha}{x'}$; this equation then becomes

$$\int_0^\infty x^{-n-1}e^{-x^2-\frac{\alpha^2}{x^2}}dx=\tfrac{1}{2}\Gamma(\tfrac{1}{2}n)\alpha^{-n}\mathrm{M}+\phi(n)\mathrm{N},$$

whence, changing the sign of n,

$$\int_0^\infty x^{n-1}e^{-x^2-\frac{\alpha^2}{x^2}}dx=\tfrac{1}{2}\Gamma(-\tfrac{1}{2}n)\alpha^{n}\mathrm{N}+\phi(-n)\mathrm{M};$$

and it follows therefore that $\phi(n)=\frac{1}{2}\Gamma(-\frac{1}{2}n)$.

Thus for all values of n (except, of course, $n=$ an even integer)

$$\int_0^\infty x^{n-1}e^{-x^2-\frac{\alpha^2}{x^2}}dx=\tfrac{1}{2}\Gamma(\tfrac{1}{2}n)\mathrm{M}+\tfrac{1}{2}\Gamma(-\tfrac{1}{2}n)\alpha^{n}\mathrm{N} \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

$$= \tfrac{1}{2}\Gamma(\tfrac{1}{2}n)\left\{1-\frac{n-1}{n-1}(2\alpha)+\frac{(n-1)(n-3)}{(n-1)(n-2)}\frac{(2\alpha)^2}{2!}-\frac{(n-1)(n-3)(n-5)}{(n-1)(n-2)(n-3)}\frac{(2\alpha)^3}{3!}+\&c.\right\}e^{2\alpha}$$

$$+\tfrac{1}{2}\Gamma(-\tfrac{1}{2}n)\alpha^{n}\left\{1-\frac{n+1}{n+1}(2\alpha)+\frac{(n+1)(n+3)}{(n+1)(n+2)}\frac{(2\alpha)^2}{2!}-\frac{(n+1)(n+3)(n+5)}{(n+1)(n+2)(n+3)}\frac{(2\alpha)^3}{3!}+\&c.\right\}e^{2\alpha}$$

$$= \tfrac{1}{2}\Gamma(\tfrac{1}{2}n)\left\{1+\frac{n-1}{n-1}(2\alpha)+\frac{(n-1)(n-3)}{(n-1)(n-2)}\frac{(2\alpha)^2}{2!}+\frac{(n-1)(n-3)(n-5)}{(n-1)(n-2)(n-3)}\frac{(2\alpha)^3}{3!}+\&c.\right\}e^{-2\alpha}$$

$$+\tfrac{1}{2}\Gamma(-\tfrac{1}{2}n)\alpha^{n}\left\{1+\frac{n+1}{n+1}(2\alpha)+\frac{(n+1)(n+3)}{(n+1)(n+2)}\frac{(2\alpha)^2}{2!}+\frac{(n+1)(n+3)(n+5)}{(n+1)(n+2)(n+3)}\frac{(2\alpha)^3}{3!}+\&c.\right\}e^{-2\alpha},$$

the series extending to infinity in every case.

The method by which the fundamental formula (7) has been obtained is open to some objections. These will be noticed, and a complete proof of (7) given, in art. 28.

21. We have

$$\frac{\Gamma(-\frac{1}{2}n)}{\Gamma(\frac{1}{2}n)}=-\frac{2}{n}\frac{\Gamma(1-\frac{1}{2}n)}{\Gamma(\frac{1}{2}n)}=-\frac{2}{n}\frac{\pi}{\sin\frac{1}{2}n\pi}\frac{1}{\{\Gamma(\frac{1}{2}n)\}^2}, \text{ since } \Gamma(m)\Gamma(1-m)=\frac{\pi}{\sin m\pi};$$

and, if n is an uneven integer, this

$$=(-)^{\frac{1}{2}(n+1)}\frac{2\pi}{n}\frac{1}{\{\sqrt{\pi}.\frac{1}{2}.\frac{3}{2}.\frac{5}{2}\ldots\frac{1}{2}(n-2)\}^2}=(-)^{\frac{1}{2}(n+1)}\frac{2^n n}{1^2.3^2.5^2\ldots n^2}=g_1,$$

g_1 being as defined in art. 16, when a is put $=2$.

Now, if n is a positive uneven integer, we have, by art. 16, $U_1+g_1V_1=R_1'$; and $M+g_1N$ is equal to $U_1+g_1V_1$ when α is written for x, and α put $=2$; so that, if n is a positive uneven integer,

$$\int_0^\infty x^{n-1}e^{-x^2-\frac{\alpha^2}{x^2}}dx=\tfrac{1}{2}\Gamma(\tfrac{1}{2}n)\left\{1+\frac{n-1}{n-1}(2\alpha)+\frac{(n-1)(n-3)}{(n-1)(n-2)}\frac{(2\alpha)^2}{2!}\right.$$
$$\left.+\frac{(n-1)(n-3)(n-5)}{(n-1)(n-2)(n-3)}\frac{(2\alpha)^3}{3!}+\&c.\right\}e^{-2\alpha},$$

the series terminating at the term preceding the first term containing a zero factor in the numerator.

Transforming the integral by assuming $x=\frac{\alpha}{x'}$, we find that, if n is a negative uneven integer,

$$\int_0^\infty x^{n-1}e^{-x^2-\frac{\alpha^2}{x^2}}dx=\tfrac{1}{2}\Gamma(-\tfrac{1}{2}n)\alpha^n\left\{1+\frac{n+1}{n+1}(2\alpha)+\frac{(n+1)(n+3)}{(n+1)(n+2)}\frac{(2\alpha)^2}{2!}+\&c.\right\}e^{-2\alpha},$$

the series terminating when the first zero factor appears.

Thus, generally,

$$\int_0^\infty x^{n-1}e^{-x^2-\frac{\alpha^2}{x^2}}dx=$$

$$\tfrac{1}{2}\Gamma(\tfrac{1}{2}n)\left\{1+\frac{n-1}{n-1}(2\alpha)+\frac{(n-1)(n-3)}{(n-1)(n-2)}\frac{(2\alpha)^2}{2!}+\frac{(n-1)(n-3)(n-5)}{(n-1)(n-2)(n-3)}\frac{(2\alpha)^3}{3!}+\&c.\right\}e^{-2\alpha}$$

$$+\tfrac{1}{2}\Gamma(-\tfrac{1}{2}n)\alpha^n\left\{1+\frac{n+1}{n+1}(2\alpha)+\frac{(n+1)(n+3)}{(n+1)(n+2)}\frac{(2\alpha)^2}{2!}+\frac{(n+1)(n+3)(n+5)}{(n+1)(n+2)(n+3)}\frac{(2\alpha)^3}{3!}+\&c.\right\}e^{-2\alpha}. \quad (8),$$

if n is not equal to an integer: but if n = a positive uneven integer, the first series continued up to the first term containing a zero factor in the numerator is the value of the integral, the second series being ignored altogether; and if n = a negative uneven integer, the second series continued up to the first term containing a zero factor in the numerator, is the value of the integral, the first series being ignored altogether. The rule may therefore be stated as follows: if neither series terminates then (8) represents the value of the integral, but if one of the series terminates, the finite series represents the value of the integral, the other being ignored; a series being supposed to terminate at the term preceding the first term that contains a zero factor in the numerator.

The apparent change of form is curious, but the reason for it has fully appeared in § I., arts. 3–6. In the 'British Association Report' for 1872 (Transactions of the

Sections, pp. 15–17) I gave the formula (8) with a brief indication of the method by which it had been obtained; this method is substantially the same as that just explained. As far as I know the general value of the integral had not been given before; although the value in the case of $n=$ an uneven integer has been long known. It is scarcely necessary to remark that in (8) n must not $=$ an even integer: this case is specially excepted throughout (see end of art. 1).

22. The case when $n=$ an uneven integer is included in a general formula given by CAUCHY in vol. i. of his 'Exercices des Mathématiques' (1826), pp. 54–56. He has there shown that if

$$P_{2i}=\int_0^\infty x^{2i}\phi(x)dx, \qquad Q_{2i}=\int_0^\infty x^{2i}\phi\left(x-\frac{1}{x}\right)dx,$$

i being a positive integer, and ϕ an even function, &c.

$$Q_{2i}=P_0+\frac{i(i+1)}{2!}P_2+\frac{(i-1)i(i+1)(i+2)}{4!}P_4\ldots+\frac{2i-1}{1}P_{2i-2}+P_{2i} \quad . \quad . \quad (9).$$

This is the case corresponding to $a=1$ of a formula proved by BOOLE (Philosophical Transactions, vol. 147, 1857, p. 783), viz.

$$\int_0^\infty x^{2i}\phi\left(x-\frac{a}{x}\right)dx=\Sigma_{m=0}^{m=i}\frac{(2m+1)(2m+2)\ldots(i+m)}{(i-m)!}a^{i-m}\int_0^\infty x^{2m}\phi(x)dx \quad . \quad . \quad (10).$$

BOOLE'S formula may however be deduced from CAUCHY'S; for, replacing $\phi(x)$ by $\phi(ax)$, we have

$$Q_{2i}=\int_0^\infty x^{2i}\phi\left(ax-\frac{a}{x}\right)dx,$$

and this integral, transformed by assuming $x=\frac{x'}{b}$, becomes

$$\frac{1}{b^{2i}}\int_0^\infty x^{2i}\phi\left(\frac{ax}{b}-\frac{ab}{x}\right)dx,$$

in which, if we put $a=b$ and replace a^2 by a, the expression subject to the functional sign becomes $x-\frac{a}{x}$*

* In the 'Messenger of Mathematics,' vol. ii., p. 79, I stated that CAUCHY'S proof was not applicable to the more general theorem in which $x-\frac{1}{x}$ was replaced by $x-\frac{a}{x}$. The error was corrected in a paper "On a Formula of CAUCHY'S for the Evaluation of a Class of Definite Integrals" ('Proceedings of the Cambridge Philosophical Society,' vol. iii., pp. 5–12, 1876); this paper contains also the theorem, corresponding to CAUCHY'S, in which ϕ is an uneven function.

To deduce the value of the integral $\int_0^\infty x^{2i}e^{-x^2-\frac{a^2}{x^2}}dx$ from BOOLE's formula, let $\phi(x)=e^{-x^2}$, then

$$P_{2i}=\int_0^\infty x^{2i}e^{-x^2}dx=\tfrac{1}{2}\Gamma(i+\tfrac{1}{2})=\tfrac{1}{2}\cdot\sqrt{\pi}\cdot\tfrac{1}{2}\cdot\tfrac{3}{2}\cdot\tfrac{5}{2}\cdots\frac{2i-1}{2};$$

and therefore the coefficient of a^{i-m}

$$=\frac{\sqrt{\pi}}{2}\frac{(2m+1)(2m+2)\ldots(i+m)}{(i+m)!}\tfrac{1}{2}\cdot\tfrac{3}{2}\cdots\frac{2m-1}{2}=\frac{\sqrt{\pi}}{2}\frac{(i+m)!}{(i-m)!m!}\frac{1}{2^{2m}}.$$

Thus

$$\int_0^\infty x^{2i}e^{-\left(x^2-\frac{a^2}{x^2}\right)^2}dx=\frac{\sqrt{\pi}}{2}\left\{a^i+\frac{(i+1)!}{(i-1)!1!}\frac{a^{i-1}}{2^2}+\frac{(i+2)!}{(i-2)!2!}\frac{a^{i-2}}{2^4}\cdots+\frac{(2i)!}{i!}\frac{1}{2^{2i}}\right\}\cdots\frac{(i+1)}{(i-1)},$$

whence

$$\int_0^\infty x^{2i}e^{-x^2-\frac{a^2}{x^2}}dx=\frac{\sqrt{\pi}}{2}a^i\left\{1+\frac{i(i+1)}{2}\left(\frac{1}{2a}\right)+\frac{(i-1)i(i+1)(i+2)}{2.4}\left(\frac{1}{2a}\right)^2+\&c.\right\}e^{-2a}. \quad (12),$$

the series being continued till it terminates of itself.

This formula is in effect that given by CAUCHY ('Exercices,' *loc. cit.*, p. 55) for the evaluation of the integral. It had however, as CAUCHY himself remarks, been previously published by LEGENDRE in vol. i., p. 366, of his 'Exercices de Calcul Intégral' (1811). LEGENDRE, whose method is quite different to CAUCHY'S, adds that EULER, in vol. iv., p. 415,* of his 'Institutiones Calculi Integralis' (1794) mentions the integrals

$$\int_0^\infty x^{-\frac{1}{2}}e^{-\frac{1+x^2}{2x}}dx, \qquad \int_0^\infty x^{-\frac{1}{2}}e^{-\frac{1+x^2}{2x}}dx,$$

which correspond to the particular cases $i=-1$ and $i=-2$ of the integral in (12), as apparently not admitting of evaluation by known methods; and he gives their values.

If in the series in (11) the terms be written in the reverse order, we have

$$\int_0^\infty x^{2i}e^{-x^2-\frac{a^2}{x^2}}dx=\frac{\sqrt{\pi}}{2}\frac{(2i)!}{2^{2i}i!}\left\{1+\frac{i}{2i}2^2a+\frac{i(i-1)}{2i(2i-1)}\frac{2^4a^2}{2!}\cdots+\frac{i!}{(2i)!}2^{2i}a^i\right\}e^{-2a},$$

which agrees with (8) when $n=2i+1$, since

$$\tfrac{1}{2}\Gamma(i+\tfrac{1}{2})=\frac{\sqrt{\pi}}{2}\frac{1.3\ldots(2i-1)}{2^i}=\frac{\sqrt{\pi}}{2}\frac{(2i)!}{2^{2i}i!}.$$

* 'Supplementum V.' ad tom. 1, cap. viii.

Transforming the integral in (12) by assuming $x=\frac{a}{x'}$, it is seen that (12) is true also when i = a negative integral; so that this formula is true when $i=$ any integer.

The same transformation shows that in general, if ϕ is an even function,

$$\int_0^\infty x^{-2i-2}\phi\left(x-\frac{a}{x}\right)dx=a^{-2i-1}\int_0^\infty x^{2i}\phi\left(x-\frac{a}{x}\right)dx.$$

Putting $2i=n-1$, the formula (12) assumes the form

$$\int_0^\infty x^{n-1}e^{-x^2-\frac{a^2}{x^2}}dx=\frac{\sqrt{\pi}}{2}a^{\frac{1}{2}(n-1)}\left\{1+(n^2-1^2)\left(\frac{1}{16a}\right)+\frac{(n^2-1^2)(n^2-3^2)}{2!}\left(\frac{1}{16a}\right)^2+\&c.\right\}e^{-2a},$$

which is true when $n=$ an uneven integer, the series being continued till it terminates of itself.

23. The investigations of the formulæ (9) and (10) given by CAUCHY and BOOLE are only applicable in the case of i an integer, and do not indicate what the formulæ become when i is unrestricted. A method, however, which I have employed in the 'Messenger of Mathematics' (vol. ii., 1872, pp. 78, 79) to prove BOOLE'S formula, and which depends on direct transformations of the integrals, leads to the general theorem.

We have

$$\int_0^\infty x^n\phi\left(x-\frac{a}{x}\right)dx=\int_{\sqrt{a}}^\infty x^n\phi\left(x-\frac{a}{x}\right)dx+\int_0^{\sqrt{a}}x^n\phi\left(x-\frac{a}{x}\right)dx;$$

and if we transform the second integral by assuming $x=\frac{\sqrt{a}}{x'}$, then, since ϕ is an even function, we find that the original integral

$$=\int_{\sqrt{a}}^\infty\phi\left(x-\frac{a}{x}\right)\left\{x^n+\frac{a^{n+1}}{x^{n+2}}\right\}dx$$

$$=\frac{1}{n+1}\int_{\sqrt{a}}^\infty\phi\left(x-\frac{a}{x}\right)\frac{d}{dx}\left[x^{n+1}-\frac{a^{n+1}}{x^{n+1}}\right]dx.$$

Now transform this integral by assuming $x-\frac{a}{x}=v$; we thus have $x=\frac{1}{2}\{v\pm\sqrt{(v^2+4a)}\}$, and, taking the upper sign, the integral becomes

$$\frac{1}{n+1}\int_0^\infty\phi(v)\frac{d}{dv}\left[\left\{\frac{v+\sqrt{(v^2+4a)}}{2}\right\}^{n+1}-\left\{\frac{-v+\sqrt{(v^2+4a)}}{2}\right\}^{n+1}\right]dv.$$

If n is an even integer, the quantity in square brackets

$$=\left\{\frac{v+\sqrt{(v^2+4a)}}{2}\right\}^{n+1}+\left\{\frac{v-\sqrt{(v^2+4a)}}{2}\right\}^{n+1}$$

$$=v^{n+1}\left\{1+(n+1)\frac{a}{v^2}+(n+1)\frac{n-2}{2!}\frac{a^2}{v^4}+(n+1)\frac{(n-3)(n-4)}{3!}\frac{a^3}{v^6}\cdots+(n+1)\frac{a^{\frac{1}{2}n}}{v^n}\right\},$$

this expression containing $\frac{1}{2}n+1$ terms.

Thus, if n is an even integer, we have

$$\int_0^\infty x^n\phi\left(x-\frac{a}{x}\right)dx=\int_0^\infty \phi(v)\left\{v^n+(n-1)av^{n-2}+\frac{(n-2)(n-3)}{2!}a^2v^{n-4}\ldots+a^{1n}\right\}dv,$$

which agrees with BOOLE's formula. But if n is not an even integer, the expression in square brackets when expanded contains an infinite number of terms, and putting, as before,

$$P_n=\int_0^\infty x^n\phi(x)dx, \qquad Q_n=\int_0^\infty x^n\phi\left(x-\frac{a}{x}\right)dx,$$

the general formula is

$$Q_n=P_n+(n-1)aP_{n-2}+\frac{(n-2)(n-3)}{2!}a^2P_{n-4}+\frac{(n-3)(n-4)(n-5)}{3!}a^3P_{n-6}+\&c.\ ad\ inf.$$

$$+a^{n+1}P_{-n-2}+(n+3)a^{n+2}P_{-n-4}+\frac{(n+4)(n+5)}{2!}a^{n+3}P_{-n-6}+\&c.\ ad\ inf.$$

24. This formula involves infinite terms unless ϕ is such a function that the integrals P_{n-2}, P_{n-4}, ... P_{-n-2}, P_{-n-6} ... are all finite. This condition is not fulfilled when $\phi(x)=e^{-x^2}$, for $\int_0^\infty x^n e^{-x^2}dx$ is infinite when $n=$ or <-1, so that we do not obtain by means of the formula a demonstration of the equation (8). If however we replace $\int_0^\infty x^n e^{-x^2}dx$ by $\Gamma\left(\frac{n+1}{2}\right)$ in all the terms, whether the integral be really infinite or not, we do in fact, as we should expect, obtain (8). For, putting $\phi(x)=e^{-x^2}$, substituting gamma-functions for the integrals, and writing $n-1$ in place of n, the formula gives

$$\int_0^\infty x^{n-1}e^{-x^2-\frac{a^2}{x^2}}dx=\tfrac{1}{2}\left\{\Gamma(\tfrac{1}{2}n)+(n-2)a\Gamma(\tfrac{1}{2}n-1)+\frac{(n-3)(n-4)}{2!}a^2\Gamma(\tfrac{1}{2}n-2)+\&c.\right\}e^{-2a}$$

$$+\tfrac{1}{2}a^n\left\{\Gamma(-\tfrac{1}{2}n)-(n+2)a\Gamma(-\tfrac{1}{2}n-1)+\frac{(n+3)(n+4)}{2!}a^2\Gamma(-\tfrac{1}{2}n-2)+\&c.\right\}e^{-2a}$$

$$=\tfrac{1}{2}\Gamma(\tfrac{1}{2}n)\left\{1+\frac{n-2}{n-2}(2a)+\frac{(n-3)(n-4)}{(n-2)(n-4)}\frac{(2a)^2}{2!}+\frac{(n-4)(n-5)(n-6)}{(n-2)(n-4)(n-6)}\frac{(2a)^3}{3!}+\&c.\right\}e^{-2a}$$

$$+\tfrac{1}{2}\Gamma(-\tfrac{1}{2}n)a^n\left\{1+\frac{n+2}{n+2}(2a)+\frac{(n+3)(n+4)}{(n+2)(n+4)}\frac{(2a)^2}{2!}+\frac{(n+4)(n+5)(n+6)}{(n+2)(n+4)(n+6)}\frac{(2a)^3}{3!}+\&c.\right\}e^{-2a}.$$

The coefficients are readily identified with those in (8), for evidently

$$\frac{(n+r+1)(n+r+2)\ldots(n+2r)}{(n+2)(n+4)\ldots(n+2r)}=\frac{(n+1)(n+3)\ldots(n+2r-1)}{(n+1)(n+2)\ldots(n+r)}.$$

This process, regarded as a method of obtaining the formula (8), is of course unsound,

and could not be rendered satisfactory without careful discussion and development. Such substitutions, however, very frequently give correct results, and it is generally interesting to examine whether, in any case that arises, the result so derived is true or not. In this instance also we thus obtain two new forms of the expression forming the right hand member of (8).

25. Transforming (8) by the assumption $x=v^{\frac{1}{m}}$, putting $m=\frac{2}{n}$, and writing a for α, it will be found that

$$\int_0^\infty e^{-v^m-\frac{a^2}{v^m}}dv=\Gamma\left(1+\frac{1}{m}\right)\left\{1+\frac{m-2}{m-2}(2a)+\frac{(m-2)(3m-2)}{(m-2)(2m-2)}\frac{(2a)^2}{2!}+\&c.\right\}e^{-2a}$$
$$-\Gamma\left(1-\frac{1}{m}\right)a^{\frac{2}{m}}\left\{1+\frac{m+2}{m+2}(2a)+\frac{(m+2)(3m+2)}{(m+2)(2m+2)}\frac{(2a)^2}{2!}+\&c.\right\}e^{-2a}. \quad . \quad (13);$$

where, as before, if either series terminates through the presence of a zero factor in a numerator, the terminating series represents the value of the integral, and if neither series terminates, both are to be included. When one of the series terminates, that is, when m = twice the reciprocal of an uneven integer, the formula may, by taking the terms of the series in the reverse order, be written

$$\int_0^\infty e^{-v^m-\frac{a^2}{v^m}}dv=\frac{\sqrt{\pi}}{m}a^{\frac{1}{m}-\frac{1}{2}}\left\{1-\frac{m^2-2^2}{m^2}\left(\frac{1}{16a}\right)+\frac{(m^2-2^2)(3^2m^2-2^2)}{m^2.2m^2}\left(\frac{1}{16a}\right)^2-\&c.\right\}e^{-2a}.$$

Putting $v=\alpha^{\frac{2}{m}}x$ and $a^2=\alpha^2\beta^2$, (13) becomes

$$\int_0^\infty e^{-\alpha^2x^m-\frac{\beta^2}{x^m}}dx=\Gamma\left(1+\frac{1}{m}\right)\alpha^{-\frac{2}{m}}\left\{1+\frac{m-2}{m-2}(2\alpha\beta)+\frac{(m-2)(3m-2)}{(m-2)(2m-2)}\frac{(2\alpha\beta)^2}{2!}+\&c.\right\}e^{-2\alpha\beta}$$
$$-\Gamma\left(1-\frac{1}{m}\right)\beta^{\frac{2}{m}}\left\{1+\frac{m+2}{m+2}(2\alpha\beta)+\frac{(m+2)(3m+2)}{(m+2)(2m+2)}\frac{(2\alpha\beta)^2}{2!}+\&c.\right\}e^{-2\alpha\beta}.$$

If for example $m=2$, we have the well-known result

$$\int_0^\infty e^{-\alpha^2x^2-\frac{\beta^2}{x^2}}dx=\Gamma\left(\tfrac{3}{2}\right)\alpha^{-1}e^{-2\alpha\beta}=\frac{\sqrt{\pi}}{2\alpha}e^{-2\alpha\beta}.$$

26. The definite integral

$$\int_0^\infty\frac{\cos bx}{(a^2+x^2)^n}dx$$

has been evaluated when n is a positive integer,* the formula in this case being

$$\int_0^\infty\frac{\cos bx}{(a^2+x^2)^n}dx=\frac{\pi}{2^n(n-1)!}\frac{b^{n-1}}{a^n}\left\{1+\frac{n(n-1)}{2}\left(\frac{1}{ab}\right)+\frac{(n+1)n(n-1)(n-2)}{2.4}\left(\frac{1}{ab}\right)^2+\&c.\right\}e^{-ab} \quad (14).$$

* See Schlömilch, 'Analytische Studien' (Leipzig, 1848), part ii., p. 97, and Crelle's Journal, vol. xxxiii., p. 273, or Catalan, 'Liouville's Journal,' ser. 1, vol. v., p. 110.

This result may be readily obtained by differentiating both members of the equation

$$\int_0^\infty \frac{\cos bx}{(a^2+x^2)^n}\,dx=\frac{\pi}{2a}\,e^{-ab}$$

$n-1$ times with regard to a^2: see *infra*, art. 31. I now proceed to investigate the value of the integral when n is unrestricted: it is to be observed, however, that n must be positive and greater than unity, for otherwise the integral is infinite in value.

It is easy to prove that the integral

$$u=x^p\int_0^\infty \frac{\cos a\xi}{(x^2+\xi^2)^p}\,d\xi$$

satisfies the differential equation

$$\frac{d^2u}{dx^2}-a^2u=\frac{p(p-1)}{x^2}\,u;$$

for, by actual differentiation,

$$\frac{d^2u}{dx^2}-\frac{p(p-1)}{x^2}u=2px^p\int_0^\infty (x^2-\xi^2-2p\xi^2)\frac{\cos a\xi}{(x^2+\xi^2)^{p+2}}\,d\xi;$$

and by a double integration by parts we find that

$$x^p\int_0^\infty \frac{\cos a\xi}{(x^2+\xi^2)^p}\,d\xi=\frac{2p}{a^2}\,x^p\int_0^\infty (x^2-\xi^2-2p\xi^2)\frac{\cos a\xi}{(x^2+\xi^2)^{p+2}}\,d\xi.*$$

Thus the value of the integral $\int_0^\infty \frac{\cos a\xi}{(x^2+\xi^2)^{p+1}}\,d\xi$ must be of the form $x^{-p-1}(AU+BV)$, where U and V are as defined in art. 3, and A and B are constants to be determined. It is however more convenient to avoid the determination of the constants by deducing the value of the integral from the formula (8) of art. 21.

In the 'Journal de l'École Polytechnique,' Cah. xvi. (vol. ix.), p. 241, Poisson has proved a formula which, after some unimportant transformations, may be written

$$\int_0^\infty x^{2n}e^{-x^2-\frac{b^2}{x^2}}\,dx=\frac{\Gamma(n+1)}{\sqrt{\pi}}\int_0^\infty \frac{\cos 2bx}{(1+x^2)^{n+1}}\,dx;$$

Poisson's demonstration holds good for all values of n such that the integral upon the right-hand side of the equation is finite. Putting $n-1$ for n and transforming the right-hand integral by assuming $x=\frac{x'}{a}$, this equation becomes

* 'Quarterly Journal of Mathematics,' vol. xii., p. 130.

$$\int_0^\infty x^{2n-2} e^{-x^2-\frac{b^2}{x^2}}\,dx = a^{2n-1}\frac{\Gamma(n)}{\sqrt{\pi}}\int_0^\infty \frac{\cos\left(\frac{2bx}{a}\right)}{(a^2+x^2)^n}\,dx\,;$$

whence, replacing $\frac{2b}{a}$ by b,

$$\int_0^\infty \frac{\cos bx}{(a^2+x^2)^n}\,dx = \frac{\sqrt{\pi}}{\Gamma(n)}a^{-2n+1}\int_0^\infty x^{2n-2}\; e^{-x^2-\frac{a^2b^2}{4x^2}}\,dx \quad . \quad . \quad . \quad . \quad . \quad (15)$$

$$= \frac{\sqrt{\pi}}{\Gamma(n)}a^{-2n+1}\left[\tfrac{1}{2}\Gamma(n-\tfrac{1}{2})\left\{1+\frac{2n-2}{2n-2}ab+\frac{(2n-2)(2n-4)}{(2n-2)(2n-3)}\frac{(ab)^2}{2!}\right.\right.$$
$$\left. + \frac{(2n-2)(2n-4)(2n-6)}{(2n-2)(2n-3)(2n-4)}\frac{(ab)^3}{3!}+\&c.\right\}$$
$$+\tfrac{1}{2}\Gamma(-n+\tfrac{1}{2})\left(\frac{ab}{2}\right)^{2n-1}\left\{1+\frac{2n}{2n}ab+\frac{2n(2n+2)}{2n(2n+1)}\frac{(ab)^2}{2!}\right.$$
$$\left.\left. + \frac{2n(2n+2)(2n+4)}{2n(2n+1)(2n+2)}\frac{(ab)^3}{3!}+\&c.\right\}\right]e^{-ab}$$

$$= \tfrac{1}{2}\frac{\sqrt{\pi}}{\Gamma(n)}\left[\Gamma(n-\tfrac{1}{2})a^{-2n+1}\left\{1+\frac{2n-2}{2n-2}ab+\frac{(2n-2)(2n-4)}{(2n-2)(2n-3)}\frac{(ab)^2}{2!}+\&c.\right\}\right.$$
$$\left.+\Gamma(-n+\tfrac{1}{2})(\tfrac{1}{2}b)^{2n-1}\left\{1+\frac{2n}{2n}ab+\frac{2n(2n+2)}{2n(2n+1)}\frac{(ab)^2}{2!}+\&c.\right\}\right]e^{-ab},$$

which represents the value of the integral for all values of n greater than unity. If n is a positive integer the first series terminates through the presence of a zero factor in a numerator, and this finite series is the value of the integral, the second series being ignored.

If n is a positive integer, then, writing the terms of the series in the reverse order,

$$\int_0^\infty \frac{\cos bx}{(a^2+x^2)^n}\,dx = \tfrac{1}{2}\frac{\sqrt{\pi}}{\Gamma(n)}\Gamma(n-\tfrac{1}{2})a^{-2n+1}\left\{\frac{(n-1)!}{(2n-2)\dots n}\frac{(2ab)^{n-1}}{(n-1)!}\right.$$
$$\left.+\frac{(n-1)\dots 2}{(2n-2)\dots(n+1)}\frac{(2ab)^{n-2}}{(n-2)!}\dots+\frac{n-1}{2n-2}(2ab)+1\right\}e^{-ab}$$

$$= \frac{\sqrt{\pi}}{\Gamma(n)}\frac{b^{n-1}}{a^n}\frac{(n-\frac{3}{2})(n-\frac{5}{2})\dots\frac{1}{2}\sqrt{\pi}}{(2n-2)\dots n}2^{n-2}\left\{1+n(n-1)\left(\frac{1}{2ab}\right)\right.$$
$$\left.+\frac{(n+1)n(n-1)(n-2)}{2!}\left(\frac{1}{2ab}\right)^2\dots+\frac{(2n-2)!}{(n-1)!}\left(\frac{1}{2ab}\right)^{n-1}\right\}e^{-ab}$$

$$= \frac{\pi}{\Gamma(n)}\frac{1}{2^n}\frac{b^{n-1}}{a^n}\left\{1+\frac{n(n-1)}{2}\left(\frac{1}{ab}\right)+\frac{(n+1)n(n-1)(n-}{2.4}\left(\frac{1}{ab}\right)^2+\&c.\right\}e^{-ab},$$

which agrees with (14).

27. In a similar manner we may obtain the value of the integral

$$\int_0^\infty \frac{x \sin bx}{(a^2+x^2)^n} dx$$

which is finite for all values of n greater than unity. For, differentiating (15) with respect to b, we have

$$\int_0^\infty \frac{x \sin bx}{(a^2+x^2)^n} dx = \tfrac{1}{2}\frac{\sqrt{\pi}}{\Gamma(n)} a^{-2n+3} b \int_0^\infty x^{2n-4} e^{-x^2-\frac{a^2b^2}{4x^2}} dx \quad \ldots\ldots (16)$$

$$= \tfrac{1}{4}\frac{\sqrt{\pi}}{\Gamma(n)} b \Big[\Gamma(n-\tfrac{3}{2}) a^{-2n+3} \Big\{ 1 + \frac{2n-4}{2n-4} ab + \frac{(2n-4)(2n-6)}{(2n-4)(2n-5)} \frac{(ab)^2}{2!} + \&c. \Big\}$$

$$+ \Gamma(-n+\tfrac{3}{2}) (\tfrac{1}{2}b)^{2n-3} \Big\{ 1 + \frac{2n-2}{2n-2} ab + \frac{(2n-2)(2n)}{(2n-2)(2n-1)} \frac{(ab)^2}{2!} + \&c. \Big\} \Big] e^{-ab},$$

where, as before, if the first series terminates, the finite portion of it represents the value of the integral.

If n is a positive integer, and the terms of the series are written in the reverse order, we find

$$\int_0^\infty \frac{x \sin bx}{(a^2+x^2)^n} dx = \frac{\pi}{2^n (n-1)!} \frac{b^{n-1}}{a^{n-1}} \Big\{ 1 + \frac{(n-1)(n-2)}{2} \Big(\frac{1}{ab}\Big) + \frac{n(n-1)(n-2)(n-3)}{2.4} \Big(\frac{1}{ab}\Big)^2 + \&c. \Big\} e^{-ab},$$

which is a known result (SCHLÖMILCH, 'Anal. Stud.,' *loc. cit.*, p. 97).

It follows at once by combining (15) and (16) that, for all values of n such that the integrals are not infinite, viz., if $n =$ or > 1,

$$\int_0^\infty \frac{x \sin bx}{(a^2+x^2)^{n+1}} dx = \frac{1}{2n} b \int_0^\infty \frac{\cos bx}{(a^2+x^2)^n} dx.$$

It would be strange if this equation were new, but I have not met with it anywhere: it is readily proved in the case of n an integer, for

$$-2a \int_0^\infty \frac{x \sin bx}{(a^2+x^2)^2} dx = \frac{d}{da} (\tfrac{1}{2}\pi e^{-ab}) = -\tfrac{1}{2}\pi b e^{-ab},$$

and

$$\int_0^\infty \frac{\cos bx}{a^2+x^2} dx = \frac{\pi}{2a} e^{-ab}$$

whence

$$\int_0^\infty \frac{x \sin bx}{(a^2+x^2)^2} dx = \tfrac{1}{2} b \int_0^\infty \frac{\cos bx}{a^2+x^2} dx,$$

which, differentiated n times with regard to a^2, gives the relation in question.

28. The method by which the formula (7) was obtained in art. 20 is not satisfactory for two reasons, (i) because the integral $\int_0^\infty x^{n-1}e^{-x}dx$ is infinite in value when n is negative, while the gamma-function, which is supposed to satisfy the equation $\Gamma(n+1)=n\Gamma(n)$ for all values of n, is finite when n is negative, except when n is a negative integer, so that we are not entitled to assume that we may always replace the integral by the gamma-function, and (ii), because it is assumed that we may change the sign of n in the equation giving the value of the integral. The following demonstration of the formula (7) is, however, I believe, quite rigorous.

The gamma-function is supposed to be defined by the equation $\Gamma(n)=\int_0^\infty x^{n-1}e^{-x}dx$ from $n=0$ to $n=1$, and by the equation $\Gamma(n+1)=n\Gamma(n)$ for all other values of n. This is in effect the definition of the gamma-function generally adopted in analysis.

We have seen in art. 20 that

$$y=\int_0^\infty x^{n-1}e^{-x^2-\frac{\alpha^2}{x^2}}dx$$

satisfies the differential equation $\frac{d^2y}{d\alpha^2}-\frac{n-1}{\alpha}\frac{dy}{d\alpha}-4y=0$, so that $y=\mathrm{A}M+\mathrm{B}\alpha^n\mathrm{N}$; and by a simple transformation of the integral, it follows that

$$\int_0^\infty x^{n-1}e^{-\alpha x^2-\frac{\beta}{x^2}}dx=\mathrm{A}\alpha^{-\frac{1}{2}n}\mathrm{H}_n+\mathrm{B}\beta^{\frac{1}{2}n}\mathrm{K}_n \quad \ldots\ldots\ldots \quad (17),$$

where

$$\mathrm{H}_n=1-\frac{1}{n-2}(2\alpha\beta)+\frac{1}{(n-2)(n-4)}\frac{(2\alpha\beta)^2}{2!}-\frac{1}{(n-2)(n-4)(n-6)}\frac{(2\alpha\beta)^3}{3!}+\&c.$$

$$\mathrm{K}_n=1+\frac{1}{n+2}(2\alpha\beta)+\frac{1}{(n+2)(n+4)}\frac{(2\alpha\beta)^2}{2!}+\frac{1}{(n+2)(n+4)(n+6)}\frac{(2\alpha\beta)^3}{3!}+\&c.$$

Suppose n to be intermediate to 0 and 1. Put $\beta=0$ in (17) and we have $\int_0^\infty x^{n-1}e^{-\alpha x^2}dx=\mathrm{A}\alpha^{-\frac{1}{2}n}$, whence $\mathrm{A}=\frac{1}{2}\Gamma(\frac{1}{2}n)$.

Now by actual differentiation of the series represented by H_n and K_n we find that

$$\frac{d}{d\alpha}(\alpha^{-\frac{1}{2}n}\mathrm{H}_n)=-\tfrac{1}{2}n\alpha^{-\frac{1}{2}n-1}\mathrm{H}_{n+2}, \qquad \frac{d\mathrm{K}_n}{d\alpha}=\frac{2\beta}{n+2}\mathrm{K}_{n+2},$$

and similarly

$$\frac{d\mathrm{H}_n}{d\beta}=-\frac{2\alpha}{n-2}\mathrm{H}_{n-2}, \qquad \frac{d}{d\beta}(\beta^{\frac{1}{2}n}\mathrm{K}_n)=\tfrac{1}{2}n\beta^{\frac{1}{2}n-1}\mathrm{K}_{n-2}.$$

Transforming the integral in (17) by assuming $x=\frac{1}{x'}$, it becomes

$$\int_0^\infty x^{-n-1}e^{-\beta x^2-\frac{\alpha}{x^2}}dx=\mathrm{A}\alpha^{-\frac{1}{2}n}\mathrm{H}_n+\mathrm{B}\beta^{\frac{1}{2}n}\mathrm{K}_n;$$

whence, differentiating with regard to β,

$$\int_0^\infty x^{-n+1}e^{-\beta x^2-\frac{\alpha}{x^2}}dx=-A\alpha^{-\frac{1}{2}n}\frac{dH_n}{d\beta}-B\frac{d}{d\beta}(\beta^{\frac{1}{2}n}K_n),$$

$$=\frac{2\alpha^{-\frac{1}{2}n+1}}{n-2}AH_{n-2}-\tfrac{1}{2}n\beta^{\frac{1}{2}n-1}BK_{n-2}.$$

Now $-n+1$ lies between 0 and 1, and putting $\alpha=0$, we have

$$\int_0^\infty x^{-n+1}e^{-\beta x^2}=\tfrac{1}{2}\beta^{\frac{1}{2}n-1}\Gamma(-\tfrac{1}{2}n+1)=-\tfrac{1}{2}n\beta^{\frac{1}{2}n-1}B,$$

giving

$$B=-\frac{1}{n}\Gamma(-\tfrac{1}{2}n+1)=\tfrac{1}{2}\Gamma(-\tfrac{1}{2}n).$$

Thus, if n lies between 0 and 1, it has been proved that

$$\int_0^\infty x^{n-1}e^{-\alpha x^2-\frac{\beta}{x^2}}dx=\tfrac{1}{2}\Gamma(\tfrac{1}{2}n)\alpha^{-\frac{1}{2}n}H_n+\tfrac{1}{2}\Gamma(-\tfrac{1}{2}n)\beta^{\frac{1}{2}n}K_n,$$

and this equation can be readily shown to be true for all values of n by differentiating both members of it any number of times with regard to α or β.

For, differentiating with regard to α,

$$\int_0^\infty x^{n+1}e^{-\alpha x^2-\frac{\beta}{x^2}}dx=\tfrac{1}{2}\Gamma(\tfrac{1}{2}n).\tfrac{1}{2}n\,\alpha^{-\frac{1}{2}n-1}H_{n+2}-\tfrac{1}{2}\Gamma(-\tfrac{1}{2}n)\quad\frac{2\beta^{\frac{1}{2}n+1}}{n+2}K_{n+2}$$

$$=\tfrac{1}{2}\Gamma(\tfrac{1}{2}n+1)\alpha^{-\frac{1}{2}n-1}H_{n+2}+\tfrac{1}{2}\Gamma(-\tfrac{1}{2}n-1)\beta^{\frac{1}{2}n+1}K_{n+2},$$

and, differentiating with regard to β,

$$\int_0^\infty x^{n-3}e^{-\alpha x^2-\frac{\beta}{x^2}}dx=\tfrac{1}{2}\Gamma(\tfrac{1}{2}n)\quad\frac{2\alpha^{-\frac{1}{2}n+1}}{n-2}H_{n-2}-\tfrac{1}{2}\Gamma(-\tfrac{1}{2}n)\tfrac{1}{2}n\beta^{\frac{1}{2}n-1}K_{n-2}$$

$$=\tfrac{1}{2}\Gamma(\tfrac{1}{2}n-1)\alpha^{-\frac{1}{2}n+1}H_{n-2}+\tfrac{1}{2}\Gamma(-\tfrac{1}{2}n+1)\beta^{\frac{1}{2}n-1}K_{n-2}.$$

If, therefore, the formula (7) is true when $n=r$, it is true when $n=r\pm1$; and it has been proved to be true for all values of n between 0 and 1: it is therefore true for all real values of n.

It may be remarked that whatever value n may have, the integral is never infinite: so that the differentiations with regard to α or β are always permissible.

§ VI.

Symbolic forms of the particular integrals in the cases in which the differential equations admit of integration in a finite form. Arts. 29–42.

29. It has been shown in art. 26 that

$$u=x^{p+1}\int_0^\infty \frac{\cos a\xi}{(x^2+\xi^2)^{p+1}}d\xi \quad \ldots\ldots\ldots \quad (18)$$

satisfies the differential equation

$$\frac{d^2u}{dx^2}-\frac{2}{x}\frac{dy}{dx}-\frac{p(p+1)}{x^2}u \quad \ldots\ldots\ldots \quad (1).$$

Now

$$\int_0^\infty \frac{\cos a\xi}{(x^2+\xi^2)^{p+1}}\,d\xi=-\tfrac{1}{2}\frac{p}{x}\frac{d}{dx}\int_0^\infty \frac{\cos a\xi}{(x^2+\xi^2)^p}\,d\xi,$$

and therefore, if p is a positive integer,

$$\int_0^\infty \frac{\cos a\xi}{(x^2+\xi^2)^{p+1}}\,d\xi=(-)^p.p!\,\frac{1}{2^p}\left(\frac{1}{x}\frac{d}{dx}\right)^p\int_0^\infty\frac{\cos a\xi}{x^2+\xi^2}d\xi$$

$$=(-)^p.p!\,\frac{\pi}{2^{p+1}}\left(\frac{1}{x}\frac{d}{dx}\right)^p\frac{e^{-ax}}{x}.$$

The complete integral of (1) is therefore

$$u=x^{p+1}\left(\frac{1}{x}\frac{d}{dx}\right)^p\left(\frac{c_1e^{ax}+c_2e^{-ax}}{x}\right) \quad \ldots\ldots\ldots \quad (19);$$

and, since $\frac{1}{x}\frac{d}{dx}e^{ax}=\frac{ae^{ax}}{x}$, this result may be written also

$$u=x^{p+1}\left(\frac{1}{x}\frac{d}{dx}\right)^{p+1}(c_1e^{ax}+c_2e^{-ax}) \quad \ldots\ldots\ldots \quad (20).$$

Since the differential equation (1) remains unaltered if $-p-1$ is substituted for p, it follows that the complete integral of (1) may be expressed also in the forms

$$u=x^{-p}\left(\frac{1}{x}\frac{d}{dx}\right)^{-p-1}\left(\frac{c_1e^{ax}+c_2e^{-ax}}{x}\right),$$

and

$$u=x^{-p}\left(\frac{1}{x}\frac{d}{dx}\right)^{-p}(c_1e^{ax}+c_2e^{-ax}).$$

30. Putting $u=x^{-p}v$, we see that the complete integral of the differential equation

$$\frac{d^2v}{dx^2}-\frac{2p}{x}\frac{dv}{dx}-a^2v=0,$$

when p is an integer, either positive or negative, is given by any one of the formulæ

$$v=x^{2p+1}\left(\frac{1}{x}\frac{d}{dx}\right)^{p+1}(c_1e^{ax}+c_2e^{-ax}),$$

$$v=\left(\frac{1}{x}\frac{d}{dx}\right)^{-p}(c_1e^{ax}+c_2e^{-ax}),$$

$$v=x^{2p+1}\left(\frac{1}{x}\frac{d}{dx}\right)^{p}\left(\frac{c_1e^{ax}+c_2e^{-ax}}{x}\right),$$

$$v=\left(\frac{1}{x}\frac{d}{dx}\right)^{-p-1}\left(\frac{c_1e^{ax}+c_2e^{-ax}}{x}\right).$$

Putting now $x=nz^{\frac{1}{n}}$, where $n=2p+1$ and $q=\frac{1}{n}$, the differential equation becomes

$$\frac{d^2v}{dz^2}-a^2z^{2q-2}v=0 \quad \ldots\ldots\ldots\ldots \quad (4);$$

and the integrals take the forms

$$v=z\left(z^{-2q+1}\frac{d}{dz}\right)^{p+1}(c_1e^{\frac{a}{q}z^q}+c_2e^{-\frac{a}{q}z^q}),$$

$$v=\left(z^{-2q+1}\frac{d}{dz}\right)^{-p}(c_1e^{\frac{a}{q}z^q}+c_2e^{-\frac{a}{q}z^q}),$$

$$v=z\left(z^{-2q+1}\frac{d}{dz}\right)^{p}\left(\frac{c_1e^{\frac{a}{q}z^q}+c_2e^{-\frac{a}{q}z^q}}{z^q}\right),$$

$$v=\left(z^{-2q+1}\frac{d}{dz}\right)^{-p-1}\left(\frac{c_1e^{\frac{a}{q}z^q}+c_2e^{-\frac{a}{q}z^q}}{z^q}\right).$$

If p is a positive integer $=i$, so that $q=\frac{1}{2i+1}$, then, from the first and third forms,

$$v=z\left(z^{-2q+1}\frac{d}{dz}\right)^{i+1}(c_1e^{\frac{a}{q}z^q}+c_2e^{-\frac{a}{q}z^q}),$$

or

$$v=z\left(z^{-2q+1}\frac{d}{dz}\right)^{i}\left(\frac{c_1e^{\frac{a}{q}z^q}+c_2e^{-\frac{a}{q}z^q}}{z^q}\right);$$

and, if p is a negative integer $=-i-1$, so that $q=-\frac{1}{2i+1}$, then, from the second and fourth forms,

$$v=\left(z^{-2q+1}\frac{d}{dz}\right)^{i+1}\left(c_1e^{\frac{a}{q}z^q}+c_2e^{-\frac{a}{q}z^q}\right),$$

or

$$v=\left(z^{-2q+1}\frac{d}{dz}\right)^{i}\left(\frac{c_1e^{\frac{a}{q}z^q}+c_2e^{-\frac{a}{q}z^q}}{z^q}\right).$$

31. These formulæ may be readily connected with the series-integrals found in § I., for, comparing (20) with the series P in arts. 3 and 5, we see that

$$x^{i+1}\left(\frac{1}{x}\frac{d}{dx}\right)^{i+1}e^{-ax}=\mathrm{A}x^{-i}\left\{1-\frac{i}{i}ax+\frac{i(i-1)}{i(i-\frac{1}{2})}\frac{a^2x^2}{2!}-\frac{i(i-1)(i-2)}{i(i-\frac{1}{2})(i-1)}\frac{a^3x^3}{3!}+\&c.\right\}e^{ax},$$

where A is a constant. Putting $i-1$ in place of i, this equation becomes

$$\left(\frac{1}{x}\frac{d}{dx}\right)^{i}e^{ax}=\mathrm{A}x^{-2i+1}\left\{1-\frac{i-1}{i-1}ax+\frac{(i-1)(i-2)}{(i-1)(i-\frac{3}{2})}\frac{a^2x^2}{2!}-\frac{(i-1)(i-2)(i-3)}{(i-1)(i-\frac{3}{2})(i-2)}\frac{a^3x^3}{3!}+\&c.\right\}e^{ax};$$

and, observing that the coefficient of $x^{-i}e^{ax}$ is a^i, it is evident that

$$\mathrm{A}=(-)^{i-1}\frac{i(i+1)\dots(2i-2)}{2^{i-1}}a.$$

Writing the terms in the reverse order, as in § IV., we find

$$\left(\frac{1}{x}\frac{d}{dx}\right)^{i}e^{ax}=\left(\frac{a}{x}\right)^{i}\left\{1-\frac{i(i-1)}{2}\frac{1}{ax}+\frac{(i+1)i(i-1)(i-2)}{2.4}\frac{1}{a^2x^2}-\&c.\right\}e^{ax}. \quad . \quad (21);$$

that is, on replacing x by $\sqrt{x}$,

$$\left(\frac{d}{dx}\right)^{i}e^{a\sqrt{x}}=\left(\frac{a}{2\sqrt{x}}\right)^{i}\left\{1-\frac{i(i-1)}{2}\frac{1}{a\sqrt{x}}+\frac{(i+1)i(i-1)(i-2)}{2.4}\frac{1}{a^2x}-\&c.\right\}e^{a\sqrt{x}},$$

which is a known formula (see, for example, SCHLÖMILCH's 'Analytische Studien' (1848), p. 86).

The formulæ which result from comparing the solutions of RICCATI's equation (4) in arts. 17 and 30 are

$$z\left(z^{-2q+1}\frac{d}{dz}\right)^{i+1}e^{\frac{a}{q}z^q}=$$

$$(-)^i\frac{q(q-1)2q(2q-1)\dots iq(iq-1)}{(q-1)(3q-1)\dots\{(2i-1)q-1\}}a\left\{1-\frac{q-1}{q(q-1)}az^q+\frac{(q-1)(3q-1)}{q(q-1)2q(2q-1)}a^2z^{2q}-\&c.\right\}e^{\frac{a}{q}z^q} \quad (22),$$

if $q=\frac{1}{2i+1}$, and

$$\left(z^{-2q+1}\frac{d}{dz}\right)^{i+1} e^{\frac{a}{q}z^q} =$$

$$(-)^i \frac{q(q+1)2q(2q+1)\dots iq(iq+1)}{(q+1)(3q+1)\dots\{(2i-1)q+1\}} az\left\{1-\frac{q+1}{(q+1)q}az^q+\frac{(q+1)(3q+1)}{(q+1)q(2q+1)2q}a^2z^{2q}-\&c.\right\}e^{\frac{a}{q}z^q} \quad (23),$$

if $q=-\frac{1}{2i+1}$.

In the case of $q=\frac{1}{2i+1}$, we have $(2i+1)q-1=0$, and therefore

$$(2i-1)q-1=-\frac{4}{2i+1}, \quad (2i-3)q-1=-\frac{8}{2i+1}, \quad \dots \quad q-1=-2iq;$$

also

$$iq=\tfrac{1}{2}-\tfrac{1}{2}q, \qquad iq-1=-\tfrac{1}{2}-\tfrac{1}{2}q, \text{ so that } iq(iq-1)=\tfrac{1}{4}(q^2-1),$$

and similarly

$$(i-1)q\{(i-1)q-1\}=\tfrac{1}{4}(3^2q^2-1), \quad \&c.$$

Thus the q-coefficient which multiplies the right-hand side of (22)

$$=\frac{(q^2-1)(3^2q^2-1)(5^2q^2-1)\dots\{(2i-1)^2q^2-1\}}{(-)^{3i}q.2q.3q\dots iq};$$

and, writing the terms on the right-hand side of (22) in the reverse order, the formula becomes

$$z\left(z^{-2q+1}\frac{d}{dz}\right)^{i+1} e^{\frac{a}{q}z^q} = a^{i+1}z^{iq} \quad \left\{1+\frac{q^2-1}{q}\left(\frac{1}{8az^q}\right)+\frac{(q^2-1)(3^2q^2-1)}{q.2q}\left(\frac{1}{8az^q}\right)^2+\&c.\right\}e^{\frac{a}{q}z^q},$$

viz.

$$z\left(z^{-2q+1}\frac{d}{dz}\right)^{\frac{q+1}{2q}} e^{\frac{a}{q}z^q} = a^{\frac{q+1}{2q}}z^{\frac{1}{2}(1-q)}\left\{1+\frac{q^2-1}{q}\left(\frac{1}{8az^q}\right)+\frac{(q^2-1)(3^2q^2-1)}{q.2q}\left(\frac{1}{8az^q}\right)^2+\&c.\right\}e^{\frac{a}{q}z^q},$$

where $q=\frac{1}{2i+1}$.

Treating the formula (23) in the same manner, we find

$$\left(z^{-2q+1}\frac{d}{dz}\right)^{\frac{q-1}{2q}} e^{\frac{a}{q}z^q} = a^{\frac{q-1}{2q}}z^{\frac{1}{2}(1-q)}\left\{1+\frac{q^2-1}{q}\left(\frac{1}{8az^q}\right)+\frac{(q^2-1)(3^2q^2-1)}{q.2q}\left(\frac{1}{8az^q}\right)^2+\&c.\right\}e^{\frac{a}{q}z^q},$$

where $q=-\frac{1}{2i+1}$.

The right-hand members of these two formulæ differ from one another and from the last expression in § IV. (art. 19) only by the powers of a which occur as factors in the two former expressions.

32. It follows from the forms of u in art. 29 that

$$x^{i+1}\left(\frac{1}{x}\frac{d}{dx}\right)^{i+1}e^{ax}=\mathrm{A}x^{-i}\left(\frac{1}{x}\frac{d}{dx}\right)^{-i}e^{ax},$$

and it can be readily verified that $\mathrm{A}=a^{2i+1}$, so that we have

$$x^{2i+1}\left(\frac{1}{x}\frac{d}{dx}\right)^{i+1}e^{ax}=a^{2i+1}\left(\frac{1}{x}\frac{d}{dx}\right)^{-i}e^{ax} \quad \ldots\ldots\ldots \quad (24).$$

Transforming this result by putting $x=\frac{1}{q}z^q$ (q unrestricted), it becomes

$$z^{(2i+1)q}\left(z^{-2q+1}\frac{d}{dz}\right)^{i+1}e^{\frac{a}{q}z^q}=a^{2i+1}\left(z^{-2q+1}\frac{d}{dz}\right)^{-i}e^{\frac{a}{q}z^q}.$$

If now $q=\frac{1}{2i+1}$, this may be written

$$z\left(z^{-2q+1}\frac{d}{dz}\right)^{\frac{q+1}{2q}}e^{\frac{a}{q}z^q}=a^{\frac{1}{q}}\left(z^{-2q+1}\frac{d}{dz}\right)^{\frac{q-1}{2q}}e^{\frac{a}{q}z^q},$$

and, on putting $q=-\frac{1}{2i+1}$, we obtain the same result; so that this formula holds good whenever q is of the forms $\pm\frac{1}{2i+1}$.

It follows from this theorem and from the two formulæ at the end of the last article that

$$z\left(z^{-2q+1}\frac{d}{dz}\right)^{\frac{q+1}{2q}}e^{\frac{a}{q}z^q}=a^{\frac{1}{q}}\left(z^{-2q+1}\frac{d}{dz}\right)^{\frac{q-1}{2q}}e^{\frac{a}{q}z^q}$$

$$=a^{\frac{q+1}{2q}}z^{\frac{1}{2}(1-q)}\left\{1-\frac{1-q^2}{q}\frac{1}{8az^q}+\frac{(1-q^2)(1-3^2q^2)}{q.2q}\left(\frac{1}{8az^q}\right)^2-\&c.\right\}e^{\frac{a}{q}z^q},$$

where $q=\pm\frac{1}{2i+1}$.

The relation (24), or, as it may be written more conveniently,

$$\left(\frac{1}{x}\frac{d}{dx}\right)^{i}x^{2i+1}\left(\frac{1}{x}\frac{d}{dx}\right)^{i+1}e^{ax}=a^{2i+1}e^{ax} \quad \ldots\ldots\ldots \quad (25),$$

admits of being established as follows.

Suppose e^{ax} expanded in ascending powers of x, and consider the term in x^p: we have

$$\left(\frac{1}{x}\frac{d}{dx}\right)^{i+1}x^p=p(p-2)\ldots(p-2i)x^{p-2i-2},$$

and

$$\left(\frac{1}{x}\frac{d}{dx}\right)^i x^{2i+1}.x^{p-2i-2}=\left(\frac{1}{x}\frac{d}{dx}\right)^i x^{p-1}=(p-1)(p-3)\ldots(p-2i+1)x^{p-2i},$$

so that

$$\left(\frac{1}{x}\frac{d}{dx}\right)^i x^{2i+1}\left(\frac{1}{x}\frac{d}{dx}\right)^{i+1} x^p=p(p-1)(p-2)\ldots(p-2i)x^{p-2i}$$

$$=\left(\frac{d}{dx}\right)^{2i+1} x^p;$$

and therefore,

$$\left(\frac{1}{x}\frac{d}{dx}\right)^i x^{2i+1}\left(\frac{1}{x}\frac{d}{dx}\right)^{i+1} e^{ax}=\left(\frac{d}{dx}\right)^{2i+1} e^{ax},$$

$$=a^{2i+1}e^{ax}.$$

The preceding investigation shows also that, if $\phi(x)$ denotes any function of x, then

$$\left(\frac{1}{x}\frac{d}{dx}\right)^i x^{2i+1}\left(\frac{1}{x}\frac{d}{dx}\right)^{i+1}\phi(x)=\left(\frac{d}{dx}\right)^{2i+1}\phi(x) \quad \ldots\ldots \quad (26);$$

for this theorem has been proved to be true when $\phi(x)$ is of the form $Ax^a+Bx^b+Cx^c+\&c.$; and as it merely asserts an identical relation between the derived functions of $\phi(x)$, it must hold good universally, since the truth of such a relation could not be dependent on the fact of whether $\phi(x)$ was or was not expressible in any particular form.

33. The general property upon which the theorem (26) depends is that the symbols of operation

$$x^{1-\alpha}\frac{d}{dx}x^{\alpha}, \quad x^{1-\beta}\frac{d}{dx}x^{\beta}, \ \&c.$$

are convertible as regards order*; that is to say, operating with such symbols upon $\phi(x)$, the result is the same in whatever order the operations are performed. This is evident, for

$$x^{1-\alpha}\frac{d}{dx}x^{\alpha}.x^{p}=(p+\alpha)x^{p},$$

so that the result of the operations upon x^p, and therefore upon $\phi(x)$, is independent of the order in which they are performed.

Now the left-hand side of (25) multiplied by x^{2i+2} is

$$\left(x^{2i+1}\frac{d}{dx}x^{-2i}.x^{2i-1}\frac{d}{dx}x^{-2i+2}\ldots x^3\frac{d}{dx}x^{-2}\right)\left(x^{2i+2}\frac{d}{dx}x^{-2i-1}.x^{2i}\frac{d}{dx}x^{-2i+1}\ldots x^2\frac{d}{dx}x^{-1}\right)x\phi(x),$$

* See Cayley, 'Proceedings of the London Mathematical Society,' vol. viii. (1876), p. 51, and also 'Solutions of the Cambridge Senate-House Problems and Riders' for 1878, pp. 99, 100.

and, writing the operators in a different order, this expression

$$=x^{2i+2}\frac{d}{dx}x^{-2i-1}.x^{2i+1}\frac{d}{dx}x^{-2i}.x^{2i}\frac{d}{dx}x^{-2i+1}\ldots x\frac{d}{dx}x^{-1}.x\phi(x)$$

$$=x^{2i+2}\left(\frac{d}{dx}\right)^{2i+1}\phi(x).$$

This investigation of (25) is in effect the same as that given in the last article, but the form in which the process is presented is somewhat preferable.

Denoting for the moment the operator $x^{1+n}\frac{d}{dx}x^{-n}$ by $[n]$, then

$$[a][a+b]\ldots[a+(i-1)b]\phi(x)=x^{1+a}\frac{d}{dx}x^{-a}.x^{1+a+b}\frac{d}{dx}x^{-a-b}\ldots x^{1+a+(i-1)b}\frac{d}{dx}x^{-a-(i-1)b}.\phi(x$$

$$=x^{a-b}\left(x^{b+1}\frac{d}{dx}\right)^{i}\frac{\phi(x)}{x^{a+(i-1)b}}.$$

Also, writing the operators in the reverse order,

$$[a+(i-1)b]\ldots[a+b][a]\phi(x)=x^{1+a+(i-1)b}\frac{d}{dx}x^{-a-(i-1)b}\ldots x^{1+a+b}\frac{d}{dx}x^{-a-b}.x^{1+a}\frac{d}{dx}x^{-a}.\phi(x)$$

$$=x^{a+ib}\left(x^{-b+1}\frac{d}{dx}\right)^{i}\frac{\phi(x)}{x^{a}}.$$

Thus

$$x^{a-b}\left(x^{b+1}\frac{d}{dx}\right)^{i}\frac{\phi(x)}{x^{a+(i-1)b}}=x^{a+ib}\left(x^{-b+1}\frac{d}{dx}\right)^{i}\frac{\phi(x)}{x^{a}},$$

and therefore, replacing $\frac{\phi(x)}{x^{a+(i-1)b}}$ by $\phi(x)$,

$$\left(x^{b+1}\frac{d}{dx}\right)^{i}\phi(x)=x^{(i+1)b}\left(x^{-b+1}\frac{d}{dx}\right)^{i}x^{(i-1)b}\phi(x) \quad \ldots\ldots (27),$$

or, writing $i+1$ for i,

$$\left(x^{b+1}\frac{d}{dx}\right)^{i+1}\phi(x)=x^{(i+2)b}\left(x^{-b+1}\frac{d}{dx}\right)^{i+1}x^{ib}\phi(x) \quad \ldots\ldots (28),$$

34. Putting $b=-2$, and $\phi(x)=\frac{e^{ax}}{x}$ and e^{ax} respectively in (27) and (28), these formulæ give

$$\left(\frac{1}{x}\frac{d}{dx}\right)^{i}\frac{e^{ax}}{x}=\frac{1}{x^{2i+2}}\left(x^{2}\frac{d}{dx}\right)^{i}\frac{e^{ax}}{x^{2i-1}},$$

$$\left(\frac{1}{x}\frac{d}{dx}\right)^{i+1}e^{ax}=\frac{1}{x^{2i+4}}\left(x^{2}\frac{d}{dx}\right)^{i+1}\frac{e^{ax}}{x^{2i}};$$

whence

$$\left(\frac{1}{x}\frac{d}{dx}\right)^{i+1}e^{ax}=a\left(\frac{1}{x}\frac{d}{dx}\right)^{i}\frac{e^{ax}}{x}=\frac{1}{x^{2i+1}}\left(x^3\frac{d}{dx}\right)^{i+1}\frac{e^{ax}}{x^{2i}}=\frac{a}{x^{2i+2}}\left(x^3\frac{d}{dx}\right)^{i}\frac{e^{ax}}{x^{2i-1}}.$$

The complete integral of (1) may therefore be written also in the form

$$u=\frac{1}{x^{i+1}}\left(x^3\frac{d}{dx}\right)^{i}\left(\frac{c_1e^{ax}+c_2e^{-ax}}{x^{2i-1}}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

or in the form

$$u=\frac{1}{x^{i+2}}\left(x^3\frac{d}{dx}\right)^{i+1}\left(\frac{c_1e^{ax}+c_2e^{-ax}}{x^{2i}}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30).$$

The first of these solutions, viz. (29), is that given by Boole in the 'Philosophical Transactions' for 1844,* and in his 'Treatise on Differential Equations,' chap. xvii., Boole's process is as follows: he shows that

$$u=e^{-i\theta}(\mathrm{D}-1)(\mathrm{D}-3)\ldots(\mathrm{D}-2i+1)v$$

where

$$\mathrm{D}\text{ denotes }\frac{d}{d\theta},\qquad x=e^{\theta},\qquad v=\frac{c_1e^{ax}+c_2e^{-ax}}{x}$$

and he thence deduces that

$$u=e^{-i\theta}.e^{\theta}\mathrm{D}e^{-\theta}.e^{2\theta}\mathrm{D}e^{-2\theta}\ldots e^{(2i-1)\theta}\mathrm{D}e^{-(2i-1)\theta}v$$

$$=e^{-(i+1)\theta}(e^{2\theta}\mathrm{D})^{i}e^{-(2i-1)\theta}v=\frac{1}{x^{i+1}}\left(x^3\frac{d}{dx}\right)^{i}\left(\frac{c_1e^{ax}+c_2e^{-ax}}{x^{2i-1}}\right).$$

But if the factors are written in the reverse order, we have

$$u=e^{-i\theta}(\mathrm{D}-2i+1)\ldots(\mathrm{D}-3)(\mathrm{D}-1)v$$

$$=e^{-i\theta}.e^{(2i-1)\theta}\mathrm{D}e^{-(2i-1)\theta}\ldots e^{2\theta}\mathrm{D}e^{-2\theta}.e^{\theta}\mathrm{D}e^{-\theta}.v$$

$$=e^{(i+1)\theta}(e^{-2\theta}\mathrm{D})^{i}e^{-\theta}v$$

$$=x^{i+1}\left(\frac{1}{x}\frac{d}{dx}\right)^{i}\left(\frac{c_1e^{ax}+c_2e^{-ax}}{x}\right),$$

which is (19). Boole does not seem to have anywhere alluded to the connexion between his own form (29) and the form (19), or to have remarked that the latter was obtainable by his own method.

Putting $b=-2q$ (q unrestricted) in (27) and (28), we find

* "On a General Method in Analysis," p. 252.

$$\left(z^{-2q+1}\frac{d}{dz}\right)^i \frac{e^{\frac{a}{q}z^q}}{z^i} = \frac{1}{z^{2iq+2q}}\left(z^{2q+1}\frac{d}{dz}\right)^i \frac{e^{\frac{a}{q}z^q}}{z^{2iq-2q}},$$

$$\left(z^{-2q+1}\frac{d}{dz}\right)^{i+1} e^{\frac{a}{q}z^q} = \frac{1}{z^{2iq+4q}}\left(z^{2q+1}\frac{d}{dz}\right)^{i+1} \frac{e^{\frac{a}{q}z^q}}{z^{2iq}};$$

so that the solution of RICCATI's equation may be written also in the forms

$$u=\frac{1}{z^i}\left(z^{2q+1}\frac{d}{dz}\right)^i \left(\frac{c_1e^{\frac{a}{q}z^q}+c_2e^{-\frac{a}{q}z^q}}{z^{1-2q}}\right),$$

$$u=\frac{1}{z^{i+1}}\left(z^{2q+1}\frac{d}{dz}\right)^{i+1}\left(\frac{c_1e^{\frac{a}{q}z^q}+c_2e^{-\frac{a}{q}z^q}}{z^{1-2q}}\right),$$

if $q=\frac{1}{2i+1}$, and

$$u=z^{1-q}\left(z^{2q+1}\frac{d}{dz}\right)^i \{z^{2q+1}(c_1e^{\frac{a}{q}z^q}+c_2e^{-\frac{a}{q}z^q})\},$$

$$u=z^{1-2q}\left(z^{2q+1}\frac{d}{dz}\right)^{i+1}\{z^{q+1}(c_1e^{\frac{a}{q}z^q}+c_2e^{-\frac{a}{q}z^q})\},$$

if $q=-\frac{1}{2i+1}$.

35. BOOLE's form (29) of the solution of the equation (1) can be obtained also from the definite integral (18) in art. 29; for we have

$$u=x^{p+1}\int_0^\infty \frac{\cos a\xi}{(x^2+\xi^2)^{p+1}}d\xi = x^{-p-1}\int_0^\infty \frac{\cos a\xi}{(1+b\xi^2)^{p+1}}d\xi, \quad \text{if } b=x^{-2},$$

$$=\frac{1}{p}x^{-p-1}\left(\int_a^\infty da\right)^2\frac{d}{db}\int_0^\infty \frac{\cos a\xi}{(1+b\xi^2)^p}d\xi$$

$$=\frac{1}{p!}x^{-p-1}\left[\left(\int_a^\infty da\right)^2\frac{d}{db}\right]^p\int_0^\infty \frac{\cos a\xi}{1+b\xi^2}d\xi, \text{ when } p \text{ is a positive integer,}$$

$$=\tfrac{1}{2}\pi.\frac{1}{p!}x^{-p-1}\left[\left(\int_a^\infty da\right)^2\frac{d}{db}\right]^p\frac{e^{-\frac{a}{\sqrt{b}}}}{\sqrt{b}}$$

$$=\tfrac{1}{2}\pi.\frac{1}{p!}x^{-p-1}\left(\frac{d}{db}\right)^p\left(\int_a^\infty da\right)^{2p}\frac{e^{-\frac{a}{\sqrt{b}}}}{\sqrt{b}}$$

$$=\tfrac{1}{2}\pi.\frac{1}{p!}x^{-p-1}\left(\frac{d}{db}\right)^p\left\{b^{\frac{1}{2}(2p-1)}e^{-\frac{a}{\sqrt{b}}}\right\}$$

$$=\tfrac{1}{2}\pi.\frac{(-1)^p}{2^p.p!}x^{-p-1}\left(x^3\frac{d}{dx}\right)^p\frac{e^{-ax}}{x^{2p-1}},$$

leading to the complete integral

$$u=x^{-p-1}\left(x^3\frac{d}{dx}\right)^p\frac{c_1e^{ax}+c_2e^{-ax}}{x^{2p-1}}.$$

It will be observed that

$$\frac{d}{db}\int_0^\infty\frac{\cos a\xi}{(1+b\xi^3)^p}d\xi=-p\int_0^\infty\frac{\xi^3\cos a\xi}{(1+b\xi^3)^{p+1}}d\xi,$$

both integrals being finite for every positive integral value of p, and that the second integral when integrated with regard to a between the limits ∞ and a

$$=-p\int_0^\infty\frac{\xi\sin(\infty\,\xi)}{(1+b\xi^3)^{p+1}}d\xi+p\int_0^\infty\frac{\xi\sin(a\xi)}{(1+b\xi^3)^{p+1}}d\xi.$$

The former of these two integrals is zero, as it can be shown that $\int_0^\infty\frac{\xi\sin a\xi}{(1+b\xi^3)^{p+1}}d\xi$ diminishes as a increases, and, in the limit when a is infinite, vanishes. A similar remark applies to the second integration with regard to a. The above process does not therefore involve the assumptions, $\sin\infty=0$, $\cos\infty=0$.

36. Poisson's theorem quoted in § V., art. 20, viz. that the definite integral (5) satisfies the differential equation (6) shows that Riccati's equation

$$\frac{d^2u}{dz^2}-a^2z^{2q-2}u=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

is satisfied by the definite integral

$$u=\int_0^\infty e^{-z^{2q}-\frac{a^2z^{2q}}{4q^2x^{2q}}}dx \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (31).$$

Putting $\frac{a^2z^{2q}}{4q^2}=\alpha$, and transforming the integral by taking $x^{2q}=\alpha x'^2$, we find that

$$u=\frac{1}{q}\alpha^{\frac{1}{2q}}\int_0^\infty x^{\frac{1}{q}-1}\,e^{-\alpha x^2-\frac{1}{x^2}}dx,$$

which, if $\frac{1}{q}-1=2i$,

$$=\frac{1}{q}(-)^i\alpha^{\frac{1}{2q}}\left(\frac{d}{d\alpha}\right)^i\int_0^\infty e^{-\alpha x^2-\frac{1}{x^2}}dx$$

$$=\frac{1}{q}(-)^i\alpha^{\frac{1}{2q}}\frac{\sqrt{\pi}}{2}\left(\frac{d}{d\alpha}\right)^i\frac{e^{-2\sqrt{\alpha}}}{\sqrt{\alpha}}=\frac{1}{q}(-)^{i+1}\alpha^{\frac{1}{2q}}\frac{\sqrt{\pi}}{2}\left(\frac{d}{d\alpha}\right)^{i+1}e^{-2\sqrt{\alpha}};$$

so that the differential equation is satisfied by

$$u=z\left(\frac{d}{d\alpha}\right)^{i+1}e^{-2\sqrt{\alpha}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (32).$$

Similarly, by transforming (31) to the form,

$$u=\frac{1}{q}\int_0^\infty x^{\frac{1}{q}-1}e^{-x^{\frac{1}{q}}-\frac{a}{x^{\frac{1}{q}}}}dx,$$

we find that the differential equation is satisfied by

$$u=\left(\frac{d}{da}\right)^{i+1}e^{-2\sqrt{a}} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (33),$$

if $\frac{1}{q}-1=-2i-2$ that is, if $q=-\frac{1}{2i+1}$.

The formulæ (32) and (33), on substituting for a its value in terms of z, lead at once to the solutions given in art. 29.

This method was applied by POISSON* to show that the equation (4) is integrable when $q=\pm\frac{1}{2i+1}$.

The formula (25) of art. 32 may be easily deduced from the equation

$$\int_0^\infty e^{-ax^2-\frac{b}{x^2}}dx=\frac{\sqrt{\pi}}{2\sqrt{a}}e^{-2\sqrt{(ab)}};$$

for we have

$$\int_0^\infty x^{2i}e^{-ax^2-\frac{b}{x^2}}dx=\frac{\sqrt{\pi}}{2}\left(-\frac{d}{da}\right)^i\frac{e^{-2\sqrt{(ab)}}}{\sqrt{a}},$$

and also

$$=\frac{\sqrt{\pi}}{2}\left(-\int_\infty^b db\right)^i\frac{e^{-2\sqrt{(ab)}}}{\sqrt{a}},$$

so that

$$\left(\int_\infty^b db\right)^i e^{-2\sqrt{(ab)}}=\sqrt{a}\left(\frac{d}{da}\right)^i\frac{e^{-2\sqrt{(ab)}}}{\sqrt{a}},$$

whence†

$$\sqrt{b}\left(\int_\infty^b db\right)^i e^{-2\sqrt{(ab)}}=-\sqrt{a}\left(\frac{d}{da}\right)^{i+1}e^{-2\sqrt{(ab)}} \quad \ldots\ldots \quad (34).$$

* 'Journal de l'École Polytechnique,' vol. ix., pp. 236, 237. POISSON's investigation is reproduced in DE MORGAN's 'Differential and Integral Calculus,' pp. 703, 704. The formulæ (32) and (33) are obtained in the same manner as in the text from the integral (31), and the solutions in art. 30 are deduced from them, in a paper "On RICCATI's Equation" ('Quarterly Journal of Mathematics,' vol. xi., 1871, pp. 267–273).

† In the paper "On RICCATI's equation," referred to in the preceding note, the following two formulæ occur

$$\left\{\int^\infty d\beta\right\}^i e^{-2\sqrt{(\alpha\beta)}}=\sqrt{\alpha}\left(-\frac{d}{d\alpha}\right)^{i+1}e^{-2\sqrt{(\alpha\beta)}}, \qquad \sqrt{\alpha}\left\{\int^\infty d\alpha\right\}^{i-1}e^{-2\sqrt{(\alpha\beta)}}=(-)^{i-1}\left(\frac{d}{d\beta}\right)^i e^{-2\sqrt{(\alpha\beta)}}.$$

These are inaccurate owing to the omission of the factor $\sqrt{\beta}$ in both, and a wrong sign in the latter; when these corrections are made, both become identical with (34). The formula (34) is given, and (35) is deduced from it, in a paper "Sur une propriété de la fonction $e^{\sqrt{x}}$" ('Nouvelle Correspondance Mathématique,' vol. ii. (1876), p. 240).

Now, identically, ϕ denoting any function,

$$a^i\left(\frac{d}{da}\right)^i\phi(ab)=b^i\left(\frac{d}{db}\right)^i\phi(ab),$$

and therefore (34) may be transformed into

$$a^{i+\frac{1}{2}}\left(\int_\infty^b db\right)^i e^{-2\sqrt{(ab)}}=-b^{i+\frac{1}{2}}\left(\frac{d}{db}\right)^{i+1}e^{-2\sqrt{(ab)}},$$

which, putting $a=1$, becomes

$$\left(\int_\infty^b db\right)^i e^{-2\sqrt{b}}=-b^{i+\frac{1}{2}}\left(\frac{d}{db}\right)^{i+1}e^{-2\sqrt{b}}.$$

Therefore

$$\left(\frac{d}{db}\right)^i b^{i+\frac{1}{2}}\left(\frac{d}{db}\right)^{i+1}e^{-2\sqrt{b}}=-e^{-2\sqrt{b}};$$

replacing $e^{-2\sqrt{b}}$ by $e^{\sqrt{b}}$, this formula becomes

$$2^{2i+1}\left(\frac{d}{db}\right)^i b^{i+\frac{1}{2}}\left(\frac{d}{db}\right)^{i+1}e^{\sqrt{b}}=e^{\sqrt{b}},$$

whence, taking $b=a^2x^2$,

$$\left(\frac{1}{x}\frac{d}{dx}\right)^i x^{2i+1}\left(\frac{1}{x}\frac{d}{dx}\right)^{i+1}e^{ax}=a^{2i+1}e^{ax} \quad \ldots\ldots\ldots \quad (35).$$

As was shown in art. 32, this is a particular case of the more general formula

$$\left(\frac{1}{x}\frac{d}{dx}\right)^i x^{2i+1}\left(\frac{1}{x}\frac{d}{dx}\right)^{i+1}\phi(x)=\left(\frac{d}{dx}\right)^{2i+1}\phi(x);$$

and this formula itself admits of generalisation, as it can be proved that

$$\left(\frac{1}{x^2}\frac{d}{dx}\right)^i x^{3i+1}\left(\frac{1}{x^2}\frac{d}{dx}\right)^i x^{3i+1}\left(\frac{1}{x^2}\frac{d}{dx}\right)^{i+1}\phi(x)=\left(\frac{d}{dx}\right)^{3i+1}\phi(x),$$

$$\left(\frac{1}{x^3}\frac{d}{dx}\right)^i x^{4i+1}\left(\frac{1}{x^3}\frac{d}{dx}\right)^i x^{4i+1}\left(\frac{1}{x^3}\frac{d}{dx}\right)^i x^{4i+1}\left(\frac{1}{x^3}\frac{d}{dx}\right)^{i+1}\phi(x)=\left(\frac{d}{dx}\right)^{4i+1}\phi(x),$$

and generally, r being any positive integer,

$$\left\{\left(\frac{1}{x^{r-1}}\frac{d}{dx}\right)^i x^{ri+1}\right\}^{r-1}\left(\frac{1}{x^{r-1}}\frac{d}{dx}\right)^{i+1}\phi(x)=\left(\frac{d}{dx}\right)^{ri+1}\phi(x).$$

These formulæ are obtained in a paper "On Certain Identical Differential Relations," published in the 'Proceedings of the London Mathematical Society,' vol. viii. (1876), pp. 47–51.

37. As mentioned in art. 34, the integral

$$u=\frac{1}{x^{i+1}}\left(x^3\frac{d}{dx}\right)^i\left(\frac{c_1e^{ax}+c_2e^{ax}}{x^{2i-1}}\right)$$

of the differential equation (1) was first given by BOOLE in the 'Philosophical Transactions' for 1844.

The integral

$$u=x^{i+1}\left(\frac{1}{x}\frac{d}{dx}\right)^i\left(\frac{c_1e^{ax}+c_2e^{-ax}}{x}\right) \quad \ldots\ldots\ldots \quad (36)$$

is due to Mr. GASKIN, and was in effect given by him in a problem set in the Cambridge Senate House Examination for 1839. The problem is as follows*:

"If m be the greatest root of the equation $m^2+m=a$,

$$Cd^m_{r=n}.\left\{\frac{\cos(x\sqrt{r}+\alpha)}{x^m\sqrt{r}}\right\} \quad \text{or} \quad Cx^{m+1}\left(\int_{r=n}-\int_{r=-n}\right)(r^2-n^2)^m\cos(rx+\alpha)$$

are general values of y in the equation $d_x^2y+\left(n^2-\frac{a}{x^3}\right)y=0$ according as m is an integer or fraction: and in the first case $(d_x^2+n^2)^{m+1}u=0$ where $u=yx^m$; apply the first or third result to solve the equation

$$d_x^2y+\left(n^2-\frac{6}{x^3}\right)y=0."$$

Thus Mr. GASKIN's theorem is that the solution of

$$\frac{d^2u}{dx^2}+a^2u=\frac{p(p+1)}{x^2}u$$

is

$$u=Cx^{-p}\left(\frac{d}{dr}\right)^p\frac{\cos(x\sqrt{r}+\alpha)}{\sqrt{r}} \quad \ldots\ldots\ldots \quad (37),$$

where r is to be put equal to a^2 after the performance of the differentiations, p being a positive integer, and that in general

$$u=Cx^{p+1}\int_{-a}^{a}(r^2-a^2)^p\cos(rx+\alpha)dr \quad \ldots\ldots\ldots \quad (38),$$

p being any positive quantity.

* The problem forms the second part of Question 8 of the paper set on the afternoon of Tuesday, January 8, 1839 ('Cambridge University Calendar,' 1839, p. 319).

The form (37) is readily identified with (36), for, from (37),

$$u = Cx^{-p}\left(\frac{1}{a}\frac{d}{da}\right)^p \frac{\cos(xa+\alpha)}{a}$$

$$= Cx^{-p}.x^{2p+1}\left(\frac{1}{\xi}\frac{d}{d\xi}\right)^p \frac{\cos(\xi+\alpha)}{\xi}, \text{ if } \xi = ax,$$

$$= Cx^{p+1}\left(\frac{1}{x}\frac{d}{dx}\right)^p \frac{\cos(ax+\alpha)}{x}.$$

A method of proving the theorems contained in Mr. GASKIN's question is given in HYMERS's 'Treatise on Differential Equations, and on the Calculus of Finite Differences' (Cambridge, 1839), pp. 83–85. The result (38) is there verified by showing that

$$v = \int_{-a}^{a} (r^2 - a^2)^p \cos(xr+\alpha)dr$$

satisfies the differential equation

$$\frac{d^2v}{dx^2} + \frac{2p+2}{x}\frac{dv}{dx} + a^2v = 0;$$

and it is remarked that (37) may be verified in a similar manner by showing that

$$v = \left(\frac{d}{dr}\right)^p \frac{\cos(x\sqrt{r}+\alpha)}{\sqrt{r}},$$

r being put equal to a^2, satisfies

$$\frac{d^2v}{dx^2} - \frac{2p}{x}\frac{dv}{dx} + a^2v = 0.*$$

The integral (37) was subsequently obtained by R. LESLIE ELLIS by a different process in the 'Cambridge Mathematical Journal,'† vol. ii., p. 195 (February, 1841). A full account of ELLIS's method, with its application to the equation in question, is given in DE MORGAN's 'Differential and Integral Calculus,' pp. 701–703.

In a paper, "Remarques sur l'équation $y'' + \frac{m}{x}y' + ny = 0$" ('LIOUVILLE's Journal,' vol. xi., 1846, pp. 338–340), M. LEBESGUE proved that the integrals of the equations

$$\frac{d^2y}{dx^2} + \frac{2i}{x}\frac{dy}{dx} + ny = 0,$$

* In the second edition (1858) of HYMERS's work, only the proof that (38) satisfies the differential equation is given (p. 128), no reference being made to Mr. GASKIN's other result. An account of BOOLE's solution and method, taken from the 'Philosophical Transactions' for 1844, is however introduced on pp. 99–106.

† "On the Integration of Certain Differential Equations," pp. 169–177, 193–201.

and

$$\frac{d^2y}{dx^2}-\frac{2i}{x}\frac{dy}{dx}+ny=0$$

are respectively

$$y=\frac{1}{x}\left\{\left\{\frac{1}{x}\left[\frac{1}{x}\cdots\left(\frac{1}{x}\nu'\right)'\right]'\cdots\right\}'\right\},$$

and

$$y=x^{2i}\left\{\left\{\frac{1}{x}\left[\frac{1}{x}\cdots\left(\frac{1}{x}\nu'\right)'\right]'\cdots\right\}'\right\},$$

where $\nu=c\sin x\sqrt{n}+c_1\cos x\sqrt{n}$, and the former of the two expressions involves i differentiations and the latter $i+1$.

In the 'Philosophical Magazine'* for May, 1856, Mr. BENJAMIN WILLIAMSON obtained by a symbolic method the integrals of the differential equations

$$\left(\mathrm{D}^2-\frac{2i}{x}\mathrm{D}+a^2\right)y=0,\quad \left(\mathrm{D}^2+\frac{2(i+1)}{x}\mathrm{D}+a^2\right)y=0$$

in the respective forms

$$y=\mathrm{A}\left(\frac{d}{da}a^{-1}\right)^i\cos(ax+\alpha),\quad y=\mathrm{A}x^{-2i+1}\left(\frac{d}{da}a^{-1}\right)^i\cos(ax+\alpha),$$

and that of the equation

$$\left(\mathrm{D}^2-\frac{i(i+1)}{x^2}+a^2\right)y=0$$

in the form

$$y=\mathrm{A}x^{-i}\left(\frac{d}{da}a^{-1}\right)^n\cos(ax+\alpha);$$

and in the 'Philosophical Transactions'† for 1857 the late Professor DONKIN obtained, also by a symbolic method, the integral of this last equation in the form

$$y=x^i\left(\mathrm{D}\frac{1}{x}\right)^i(c_1\sin ax+c_2\cos ax).$$

* "On the Solution of Certain Differential Equations" ('Philosophical Magazine,' Fourth series, vol. xi., pp. 364–371).

† "On the Equation of LAPLACE's Functions, &c.," vol. 147, p. 44. A proof that the integral of the partial differential equation $\frac{1}{a^2}\frac{d^2u}{dt^2}=\frac{d^2u}{dr^2}+\frac{2}{r}\frac{du}{dr}-\frac{i(i+1)}{r^2}u$, which is a simple transformation of (1), may be presented in the form $u=r^i\left(\frac{1}{r}\frac{d}{dr}\right)^i\frac{\phi(r+at)+\psi(r-at)}{r}$ is given by Professor C. NIVEN in the 'Solutions of the Senate-House Problems and Riders' for 1878, pp. 158, 159.

38. Taking the differential equation in the form (1), which has been adopted as the standard form in this memoir, it may be observed that, although the integrals

$$u=x^i\left(\frac{d}{dx}\frac{1}{x}\right)^i(c_1e^{ax}+c_2e^{-ax})$$

and

$$u=x^{i+1}\left(\frac{1}{x}\frac{d}{dx}\right)^i\left(\frac{c_1e^{ax}+c_2e^{-ax}}{x}\right) \quad \ldots\ldots\ldots \quad (39)$$

have thus been given more than once by different mathematicians, the slightly modified form

$$u=x^{i+1}\left(\frac{1}{x}\frac{d}{dx}\right)^{i+1}(c_1e^{ax}+c_2e^{-ax})$$

seems scarcely to have been noticed.* It was this form which led me to the solution in § II. as follows: if x^2 is written for ξ after the performance of the differentiations, then

$$x^{i+1}\left(\frac{1}{x}\frac{d}{dx}\right)^{i+1}e^{ax}=2^{i+1}x^{i+1}\left(\frac{d}{d\xi}\right)^{i+1}e^{a\sqrt{\xi}}$$

$$=2^{i+1}\,.\,i!\,.\,x^{i+1}\,.\,\text{coefficient of } h^{i+1} \text{ in } e^{h\frac{d}{d\xi}}.e^{a\sqrt{\xi}}$$

$$=2^{i+1}\,.\,i!\,.\,x^{i+1}\,.\,\text{coefficient of } h^{i+1} \text{ in } e^{a\sqrt{(\xi+h)}}$$

$$=2^{i+1}\,.\,i!\,.\,x^{i+1}\,.\,\text{coefficient of } h^{i+1} \text{ in } e^{a\sqrt{(x^2+h)}}$$

$$=2^{i+1}\,.\,i!\,.\,\text{coefficient of } h^{i+1} \text{ in } e^{a\sqrt{(x^2+zh)}}.$$

In my paper "On a Differential Equation allied to RICCATI'S" ('Quarterly Journal,' vol. xii., 1872, p. 136), I deduced by this method from the form (39), which is the same as (19) of art. 29, that the solution of (1) was

$$u=x^{i+1}\,.\,\text{coefficient of } h^i \text{ in } \frac{c_1e^{a\sqrt{(x^2+h)}}+c_2e^{-a\sqrt{(x^2+h)}}}{\sqrt{(x^2+h)}},$$

but I did not then remark the far more simple form

$$u=\text{coefficient of } h^{i+1} \text{ in } c_1e^{a\sqrt{(x^2+zh)}}+c_2e^{-a\sqrt{(x^2+zh)}}.$$

39. It is interesting to connect Mr. GASKIN'S definite-integral solution (38) of art. 37 with that given in art. 26. The latter is

$$u=x^{p+1}\int_0^\infty\frac{\cos a\xi}{(x^2+\xi^2)^{p+1}}d\xi \quad \ldots\ldots\ldots \quad (40),$$

* The integral is however in effect expressed in this form in EARNSHAW'S 'Partial Differential Equations' (1871), p. 92.

p being supposed to be any positive quantity; and the process of verifying that this is a solution of the equation is as follows. By actual differentiation, we have, as in art. 26,

$$\frac{d^2u}{dx^2}-\frac{p(p+1)}{x^2}u=2(p+1)x^{p+1}\int_0^\infty\{x^2-(2p+3)\xi^2\}\frac{\cos a\xi}{(x^2+\xi^2)^{p+3}}d\xi \quad . \quad . \quad (41),$$

and, by a double integration by parts,

$$\int_0^\infty\frac{\cos a\xi}{(x^2+\xi^2)^{p+1}}d\xi=\left[\frac{1}{a}\frac{\sin a\xi}{(x^2+\xi^2)^{p+1}}-\frac{2(p+1)\xi}{a^2}\frac{\cos a\xi}{(x^2+\xi^2)^{p+2}}\right]_0^\infty$$
$$+\frac{2(p+1)}{a^2}\int_0^\infty\{x^2-(2p+3)\xi^2\}\frac{\cos a\xi}{(x^2+\xi^2)^{p+3}}d\xi \quad . \quad . \quad . \quad (42).$$

The integral (40) therefore satisfies the differential equation, since the quantity in square brackets vanishes between the limits of integration.

If these limits had been any quantities α, β independent of x, instead of 0, ∞, we should have obtained a result corresponding to (41), but the quantity in square brackets in (42) would not have vanished. Replacing $p+1$ by $-p$, it is clear that

$$u=x^{-p}\int_\alpha^\beta(x^2+\xi^2)^p\cos a\xi d\xi$$

will satisfy the differential equation if α, β can be so chosen that

$$\left[\frac{1}{a}(x^2+\xi^2)^p\sin a\xi+\frac{2p\xi}{a^2}(x^2+\xi^2)^{p-1}\cos a\xi\right]_\alpha^\beta$$

is zero. This would be the case if* $\beta=xi'$, $\alpha=-xi'$, but these values of α, β are inadmissible as they are not independent of x.

Transforming now the integral in (40) by the substitution $a\xi=xt$, we have

$$x^{p+1}\int_0^\infty\frac{\cos a\xi}{(x^2+\xi^2)^{p+1}}d\xi=a^{2p+1}x^{-p}\int_0^\infty\frac{\cos xt}{(a^2+t^2)^{p+1}}dt,$$

and therefore

$$u=x^{-p}\int_0^\infty\frac{\cos xt}{(a^2+t^2)^{p+1}}dt \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (43)$$

also satisfies the differential equation.

To verify this, we find by differentiation

$$\frac{d^2u}{dx^2}-\frac{p(p+1)}{x^2}u=2px^{-p-1}\int_0^\infty\frac{t\sin xt}{(a^2+t^2)^{p+1}}dt-x^{-p}\int_0^\infty\frac{t^2\cos xt}{(a^2+t^2)^{p+1}}dt \quad . \quad . \quad . \quad . \quad (44)$$

* As i denotes a positive integer in this memoir, i' is used to denote $\sqrt{(-1)}$.

and, by integration by parts,

$$2p\int_0^\infty \frac{t\sin xt}{(a^2+t^2)^{p+1}}dt=\left[-\frac{\sin xt}{(a^2+t^2)^p}\right]_0^\infty+\int_0^\infty \frac{x\cos xt}{(a^2+t^2)^p}dt \quad \ldots\ldots \quad (45),$$

whence the right-hand member of (44)

$$=x^{-p}\int_0^\infty \left\{\frac{\cos xt}{(a^2+t^2)^p}-\frac{t^2\cos xt}{(a^2+t^2)^{p+1}}\right\}dt$$

$$=x^{-p}a^2\int_0^\infty \frac{\cos xt}{(a^2+t^2)^{p+1}}dt=a^2u.$$

If the limits were α, β the differential equation would still be satisfied if the quantity in square brackets in (45) vanished between these limits. This is not the case for any other values of α and β besides 0 and ∞, but if in (43) $p+1$ is replaced by $-p$, so that the integral is

$$u=x^{p+1}\int_\beta^\alpha (a^2+t^2)^p\cos xt\,dt,$$

then the quantity in square brackets $=-(a^2+t^2)^{p+1}\sin xt$, which vanishes when $t=\pm a i'$, and therefore the differential equation is satisfied by the integral

$$u=x^{p+1}\int_{-ai'}^{ai'}(a^2+t^2)^p\cos xt\,dt.$$

Since in this case the quantity in square brackets vanishes in virtue of the factor $(a^2+t^2)^{p+1}$, we may replace $\cos xt$ by $\cos(xt+\alpha)$, α being any constant, so that the solution of the differential equation may be written

$$u=Cx^{p+1}\int_{-ai'}^{ai'}(a^2+t^2)^p\cos(xt+\alpha)dt \quad \ldots\ldots \quad (46).$$

If in the differential equation a^2 be replaced by $-a^2$, this integral becomes

$$u=Cx^{p+1}\int_{-a}^{a}(t^2-a^2)^p\cos(xt+\alpha)dt \quad \ldots\ldots \quad (47),$$

which is Mr. GASKIN's formula (38).

40. This is not however, as stated by Mr. GASKIN, the general integral of the differential equation, as it in fact contains only one arbitrary constant. For, evidently,

$$\int_{-a}^{a}(t^2-a^2)^p\sin xt\,dt=0,$$

so that the introduction of the constant α does not increase the generality of the solution.

Returning to the integral (46), we find, on putting $\alpha=0$ and transforming the integral by the substitution $t=ix^{-1}v$,

$$u=Cx^{p+1}\int_{-a}^{a}(v^2-a^2)^p(e^{xv}+e^{-xv})dv \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (48)$$

$$=Cx^{p+1}\left(\frac{d^2}{dx^2}-a^2\right)^p\int_{-a}^{a}(e^{xv}+e^{-xv})dv$$

$$=Cx^{p+1}\left(\frac{d^2}{dx^2}-a^2\right)^p\left(\frac{e^{ax}-e^{-ax}}{x}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (49).$$

Now, as will be shown in the next article,

$$\left(\frac{d^2}{dx^2}-a^2\right)^i\frac{e^{ax}}{x}=(-)^i2^ii!\left(\frac{1}{x}\frac{d}{dx}\right)^i\frac{e^{ax}}{x} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (50),$$

so that the particular integral (49) is equivalent to

$$u=Cx^{p+1}\left(\frac{1}{x}\frac{d}{dx}\right)^{p+1}(e^{ax}-e^{-ax}).$$

The complete solution of the differential equation is, by art. 29,

$$u=x^{p+1}\left(\frac{1}{x}\frac{d}{dx}\right)^{p+1}(c_1e^{ax}+c_2e^{-ax}),$$

which may therefore be written

$$u=x^{p+1}\left(\frac{d^2}{dx^2}-a^2\right)^p\left(\frac{c_1e^{ax}+c_2e^{-ax}}{x}\right)$$

$$=x^{p+1}\left(\frac{d^2}{dx^2}-a^2\right)^p\left(c_1\int_{-a}^{a}e^{xv}dv+c_2\int_{-a}^{a}e^{-xv}dv\right)$$

$$=x^{p+1}\left(c_1\int_{-a}^{a}(v^2-a^2)^pe^{xv}dv+c_2\int_{-a}^{a}(v^2-a^2)^pe^{-xv}dv\right).$$

This is the complete solution in the form corresponding to (48).

41. To prove the relation (50), let

$$v=\int_0^\infty e^{-a^2x^2-\frac{b^2}{x^2}}dx=\frac{\sqrt{\pi}}{2a}e^{-2ab},$$

then, denoting for the sake of brevity $a^2x^2+\frac{b^2}{x^2}$ by w,

$$\frac{dv}{da}=\int_0^\infty -2ax^2e^{-w}dx,$$

whence

$$\left(-\frac{1}{2a}\frac{d}{da}\right)v=\int_0^\infty x^2e^{-w}dx$$

and therefore

$$\left(-\frac{1}{2a}\frac{d}{da}\right)^i v=\int_0^\infty x^{2i}e^{-w}dx \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (51).$$

Again,

$$\frac{d^2v}{da^2}=\int_0^\infty(-2x^2+4a^2x^4)e^{-w}dx,$$

and

$$\int_0^\infty 4a^2x^4e^{-w}dx=\left[-e^{-a^2x^2}2x^3e^{-\frac{b^2}{x^2}}\right]_0^\infty+\int_0^\infty e^{-a^2x^2}\frac{d}{dx}(2x^3e^{-\frac{b^2}{x^2}})dx$$

$$=\int_0^\infty(6x^2+4b^2)e^{-w}dx\,;$$

therefore

$$\frac{d^2v}{da^2}=\int_0^\infty(4x^2+4b^2)e^{-w}dx,$$

whence

$$\left(\frac{d^2}{da^2}-4b^2\right)v=4\int_0^\infty x^2e^{-w}dx \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (52).$$

If instead of the integral v we start with the integral

$$v_i=\int_0^\infty x^{2i}e^{-w}dx,$$

we have

$$\frac{d^2v_i}{da^2}=\int_0^\infty(-2x^{2i+2}+4a^2x^{2i+4})e^{-w}dx,$$

and, integrating the second term by parts as before, we find

$$\frac{d^2v_i}{da^2}=\int_0^\infty\{(4i+4)x^{2i+2}+4b^2x^{2i}\}e^{-w}dx,$$

so that

$$\left(\frac{d^2}{da^2}-4b^2\right)v_i=4(i+1)\int_0^\infty x^{2i+2}e^{-w}dx=4(i+1)v_{i+1} \quad . \quad . \quad . \quad . \quad (53).$$

Thus from (52) and (53)

$$\left(\frac{d^2}{da^2}-4b^2\right)^i v=4^i.i!\int_0^\infty x^{2i}e^{-u}dx$$

$$=4^i.i!\left(-\frac{1}{2a}\frac{d}{da}\right)^i v, \text{ from (51)},$$

which is the relation (50).

It follows from (50) in connexion with (21) of art. 31 that

$$\left(\frac{d^2}{dx^2}-a^2\right)^i\frac{e^{ax}}{x}=(-)^i\frac{(2a)^i}{x^{i+1}}i!\left\{1-\frac{(i+1)i}{2}\frac{1}{ax}+\frac{(i+2)(i+1)i(i-1)}{2.4}\frac{1}{a^2x^2}-\&c.\right\}e^{ax}. \quad (54),$$

a formula given by HARGREAVE in the 'Philosophical Transactions'* for 1848, p. 34.

42. In the paper just referred to HARGREAVE obtained by a symbolic process the solution of the differential equation in the form

$$u=x^{i+1}(\mathrm{D}^2-a^2)\left(\frac{c_1e^{ax}+c_2e^{-ax}}{x}\right),$$

and thence, by (54), deduced the solution in the expanded form.

HARGREAVE also gives on p. 45 of his memoir the complete solution of the equation

$$\frac{d^2u}{dx^2}+\frac{2m}{x}-a^2x=0$$

in the form

$$u=c_1\int_{-a}^{a}(z^2-a^2)^{m-1}e^{xz}dz+c_2x^{-2m+1}\int_{-a}^{a}(z^2-a^2)^{-m}e^{xz}dz$$

$$=c_1\int_{-1}^{1}(z^2-1)^{m-1}e^{axz}dz+c_2x^{-2m+1}\int_{-1}^{1}(z^2-1)^{-m}e^{axz}dz. \quad . \quad . \quad . \quad . \quad (55).$$

One or other of the definite integrals in (55) is however always infinite, except when m lies between 0 and 1.

In the case of the differential equation (1), this solution becomes

$$u=c_1x^{-p}\int_{-1}^{1}(z^2-1)^{-p-1}e^{axz}+c_2x^{p+1}\int_{-1}^{1}(z^2-1)^pe^{axz}dz,$$

or, as it may be written more conveniently,

$$u=c_1x^{-p}\int_{-1}^{1}(1-z^2)^{-p-1}e^{axz}+c_2x^{p+1}\int_{-1}^{1}(1-z^2)^pe^{axz}dz.$$

* "On the Solution of Linear Differential Equations," pp. 31–54.

It is easy to connect these definite integrals with the series U and V of art. 3, for

$$\int_{-1}^{1}(1-z^2)^{-p-1}e^{axz}dz=\int_{-1}^{1}(1-z^2)^{-p-1}\left(1+axz+\frac{a^2x^2z^2}{2!}+\frac{a^3x^3z^3}{3!}+\&c.\right)dz$$

$$=2\int_{0}^{1}(1-z^2)^{-p-1}\left(1+\frac{a^2x^2z^2}{2!}+\frac{a^4x^4}{4!}+\&c.\right)dz$$

$$=\frac{\Gamma(-p)\Gamma(\frac{1}{2})}{\Gamma(-p+\frac{1}{2})}+\frac{a^2x^2}{2!}\frac{\Gamma(-p)\Gamma(\frac{3}{2})}{\Gamma(-p+\frac{3}{2})}+\frac{a^4x^4}{4!}\frac{\Gamma(-p)\Gamma(\frac{5}{2})}{\Gamma(-p+\frac{5}{2})}+\&c.$$

$$=\frac{\Gamma(-p)\Gamma(\frac{1}{2})}{\Gamma(-p+\frac{1}{2})}\left\{1+\frac{a^2x^2}{2^2}\frac{\frac{1}{2}}{-p+\frac{1}{2}}+\frac{a^4x^4}{4!}\frac{\frac{1}{2}\frac{3}{2}}{(-p+\frac{1}{2})(-p+\frac{3}{2})}+\&c.\right\}$$

$$=\sqrt{\pi}.\frac{\Gamma(-p)}{\Gamma(-p+\frac{1}{2})}\left\{1-\frac{1}{p-\frac{1}{2}}\frac{a^2x^2}{2^2}+\frac{1}{(p-\frac{1}{2})(p-\frac{3}{2})}\frac{a^4x^4}{2^4.2!}-\&c.\right\},$$

whence

$$x^{-p}\int_{-1}^{1}(1-z^2)^{-p-1}e^{axz}dz=\sqrt{\pi}.\frac{\Gamma(-p)}{\Gamma(-p+\frac{1}{2})}\mathrm{U}.$$

Similarly

$$x^{p+1}\int_{-1}^{1}(1-z^2)^{p}e^{axz}dz \quad =\sqrt{\pi}.\frac{\Gamma(p+1)}{\Gamma(p+\frac{3}{2})}\ \mathrm{V};$$

and we thus obtain expressions for U and V as definite integrals, taken between the limits 1 and -1, for all values of p for which the integrals are finite.

§ VII.

Connexion with BESSEL'S *Functions.* Arts. 43–48.

43. If the differential equation (1) is transformed by putting $u=x^{\frac{1}{2}}w$, it assumes the form

$$\frac{d^2w}{dx^2}+\frac{1}{x}\frac{dw}{dx}-a^2w-(p+\tfrac{1}{2})^2\frac{w}{x^2}=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (56).$$

The equation of BESSEL'S Functions is

$$\frac{d^2w}{dx^2}+\frac{1}{x}\frac{dw}{dx}+\left(1-\frac{\nu^2}{x^2}\right)w=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (57),$$

so that (56) becomes identical with (57) if

$$a=\sqrt{(-1)}=i', \qquad p+\tfrac{1}{2}=\nu.$$

We may therefore pass from the solutions of the equation (1) to the solutions of BESSEL'S equation (57) by multiplying by $x^{-\frac{1}{2}}$ and putting $a=\sqrt{(-1)}$, $p=\nu-\frac{1}{2}$.

44. The BESSEL'S Function, $\mathrm{J}^\nu(x)$, may be defined for real values of ν greater than $-\frac{1}{2}$ by either of the formulæ

$$\mathrm{J}^\nu(x)=\frac{x^\nu}{2^\nu\Gamma(\nu+1)}\left\{1-\frac{x^2}{2(2\nu+2)}+\frac{x^4}{2.4(2\nu+2)(2\nu+4)}-\&c.\right\} \quad . \quad . \quad . \quad (58),$$

$$\mathrm{J}^\nu(x)=\frac{1}{\sqrt{\pi}}\frac{x^\nu}{2^\nu\Gamma(\nu+\frac{1}{2})}\int_{-1}^{1}e^{ixu}(1-u^2)^{\nu-\frac{1}{2}}du \quad . \quad . \quad . \quad . \quad . \quad . \quad (59),$$

where i' denotes, as throughout, $\sqrt{(-1)}$.

Comparing (58) with the expression V in art. 3, we see that if $\nu=p+\frac{1}{2}$, and if a is replaced by i', the series in the two formulæ become identical, the exact relation between V and BESSEL'S Function being

$$\mathrm{V}=\mathrm{A}x^{\frac{1}{2}}\mathrm{J}^{p+\frac{1}{2}}(i'ax),$$

where A denotes the constant

$$\left(\frac{2}{i'a}\right)^{p+\frac{1}{2}}\Gamma(p+\tfrac{3}{2})$$

and p is supposed to be positive.

The formula (59) corresponds to Mr. GASKIN'S definite integral solution (38) or to one of the definite integrals in HARGREAVE'S solution (55).

45. It is known that $\mathrm{J}^\nu(x)$ may be exhibited as the sum of two series multiplied respectively by $\sin x$ and $\cos x$, viz.*

$$\mathrm{J}^\nu(x)=\frac{x^\nu}{2^\nu\Gamma(\nu+1)}(\mathrm{A}\cos x+\mathrm{B}\sin x) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (60),$$

where

$$\mathrm{A}=1-\frac{2\nu+3}{2\nu+2}\frac{x^2}{2!}+\frac{(2\nu+5)(2\nu+7)}{(2\nu+2)(2\nu+4)}\frac{x^4}{4!}-\frac{(2\nu+7)(2\nu+9)(2\nu+11)}{(2\nu+2)(2\nu+4)(2\nu+6)}\frac{x^6}{6!}+\&c.,$$

$$\mathrm{B}=x-\frac{2\nu+5}{2\nu+2}\frac{x^3}{3!}+\frac{(2\nu+7)(2\nu+9)}{(2\nu+2)(2\nu+4)}\frac{x^5}{5!}-\frac{(2\nu+9)(2\nu+11)(2\nu+13)}{(2\nu+2)(2\nu+4)(2\nu+6)}\frac{x^7}{7!}+\&c.$$

* LOMMEL'S 'Studien über die Bessel'schen Functionen' (1868), p. 17, or TODHUNTER'S 'Treatise on LAPLACE'S Functions, LAMÉ'S Functions, and BESSEL'S Functions' (1875), p. 292.

This formula may be written

$$J^\nu(x)=\frac{x^\nu}{2^{\nu+1}\Gamma(\nu+1)}\Big\{\Big(1-i'x+\frac{2\nu+3}{2\nu+2}\frac{i'^2x^2}{2!}-\frac{2\nu+5}{2\nu+2}\frac{i'^3x^3}{3!}+\&c.\Big)e^{i'x}$$

$$+\Big(1+i'x+\frac{2\nu+3}{2\nu+2}\frac{i'^2x^2}{2!}+\frac{2\nu+5}{2\nu+2}\frac{i'^3x^3}{3!}+\&c.\Big)e^{-i'x}\Big\}.$$

and the expression on the right-hand side therefore corresponds to $\frac{1}{2}(Q+S)$ where Q and S are as defined in art. 3, so that the algebraic theorem to which the two forms of Bessel's Functions (58) and (60) lead is $V=\frac{1}{2}(Q+S)$.

46. The formula involving descending series for Bessel's Function, $J^\nu(x)$ is

$$J^\nu(x)=\sqrt{\Big(\frac{2}{\pi x}\Big)}\cdot\Big\{1\quad-\frac{(4\nu^2-1^2)(4\nu-3^2)}{1.2.(8x)^2}\qquad+\&c.\Big\}\cos(x-\tfrac{1}{4}\pi-\tfrac{1}{2}\nu\pi)$$

$$-\sqrt{\Big(\frac{2}{\pi x}\Big)}\cdot\Big\{\frac{4\nu^2-1}{1.8x}-\frac{(4\nu^2-1^2)(4\nu^2-3^2)(4\nu^2-5^2)}{1.2.3(8x)^3}+\&c.\Big\}\sin(x-\tfrac{1}{4}\pi-\tfrac{1}{2}\nu\pi);$$

the descending series ultimately diverge for all values of ν for which they do not terminate, but the converging terms may be used for the calculation of $J^\nu(x)$; and this formula was in fact employed by Hansen in the calculation of his tables of $J^0(x)$ and $J^1(x)$*. If $\nu=p+\frac{1}{2}$, p being an integer, the series terminate and we obtain a finite expression for $J^{p+\frac{1}{2}}(x)$.

Replacing the sine and cosine by their exponential values, this formula may be written

$$J^\nu(x)=\frac{i'^{\nu+\frac{1}{2}}}{\sqrt{(2\pi x)}}\{(-)^{\nu+\frac{1}{2}}e^{i'x}\alpha+e^{-i'x}\beta\},$$

where

$$\alpha=1-\frac{4\nu^2-1^2}{1}\frac{1}{8i'x}+\frac{(4\nu^2-1^2)(4\nu^2-3^2)}{1.2}\frac{1}{(8i'x)^2}-\&c.$$

and β differs from α only in having all the terms positive.

* Lommel's 'Studien über die Bessel'schen Functionen,' p. 58.

Putting $\nu=p+\frac{1}{2}$, we have therefore

$$J^{p+\frac{1}{2}}(x)=\frac{1}{\sqrt{(2\pi x)}}i'^{p+1}\{(-)^{p+1}e^{i'x}\alpha_1+e^{-i'x}\beta_1\},$$

where

$$\alpha_1=1-\frac{p(p+1)}{2}\frac{1}{i'z}+\frac{(p-1)p(p+1)(p+2)}{2.4}\frac{1}{(i'x)^2}-\&c.$$

and β_1 denotes a similar series, having all the terms positive.

If p is a positive integer, this expression corresponds to $\frac{1}{2g}(R'-P')$, where the terms in R′ and P′ are written in the reverse order, as in § IV. If p is not an integer, the series, as already mentioned, are divergent, so that, strictly speaking, the formula only has a meaning when it contains a finite number of terms. An expression can however be found for the remainder after a finite number of terms, *i.e.*, for the difference between $J^{p+\frac{1}{2}}(x)$ and the sum of these terms, by means of which the use of the formula in calculation may be justified.

47. It is a known theorem in BESSEL's Functions that if p is a positive integer, $J^p(x)$ is equal to the coefficient of z^p in the expansion of $e^{\frac{x}{2}(z-\frac{1}{z})}$; and it follows therefore, by means of the relation between V and $J^{p+\frac{1}{2}}(i'ax)$ in art. 44, that, if $p+\frac{1}{2}=$ an even positive integer $=2m$,

$$V=(-)^m\frac{2^{2m}\Gamma(2m+1)}{a^{2m}}x^{\frac{1}{2}}\times \text{ coefficient of } z^{2m} \text{ in } \cos\frac{ax}{2}\left(z-\frac{1}{z}\right),$$

and if $p+\frac{1}{2}=$ an uneven positive integer $=2m+1$,

$$V=(-)^m\frac{2^{2m+1}\Gamma(2m+2)}{a^{2m+1}}x^{\frac{1}{2}}\times \text{ coefficient of } z^{2m+1} \text{ in } \sin\frac{ax}{2}\left(z-\frac{1}{z}\right).$$

48. It was shown in § II. that the differential equation (1) was satisfied by the coefficient of h^{p+1} in the expansion of $e^{x\sqrt{(x^2+xh)}}$, and we thus find that if $\nu=p+\frac{1}{2}$, p being a positive integer, the general integral of BESSEL's equation (57) is

$$w=x^{-\frac{1}{2}}\{c_1\times \text{ coefficient of } h^{p+1} \text{ in the expansion of } \cos\sqrt{(x^2+xh)}$$
$$+c_2 \quad " \quad " \quad " \quad \sin\sqrt{(x^2+xh)}\}.$$

§ VIII.

Writings specially connected with the contents of the memoir.

[When only a portion of a paper relates to the subject of the memoir, the page-numbers refer only to this portion.]

WRITINGS REFERRED TO IN §§ I., II., III.

(i.) 1868. CAYLEY. "On RICCATI's Equation." 'Philosophical Magazine,' Fourth series, vol. xxxvi., pp. 348–351.

The equation is written in the form

$$\frac{d^2u}{dx^2} = ax^{2q-2}u,$$

and the expressions P_2, Q_2, R_2, S_2 of art. 17 are obtained by assuming series of the forms in question and equating coefficients. Two of the series terminate when q is the reciprocal of an uneven integer.

(ii.) 1869. —— "Note on the Integration of Certain Differential Equations by Series." 'Messenger of Mathematics,' First series, vol. v., pp. 77–82.

It is shown that if we have a solution

$$A\left(x^\alpha + \frac{a_1}{b_1}x^{\alpha+1} + \frac{a_1a_2}{b_1b_2}x^{\alpha+2} + \&c.\right)$$

of a differential equation, and that if one of the factors in a numerator, say a_r, vanishes, then we may stop at the preceding term, the finite series so obtained being a particular integral; but that if we continue the series, notwithstanding the evanescent factor, and if at length a factor in a denominator, say $b_s (s>r)$, vanishes, then the series recommences with the term involving $x^{\alpha+s}$, and we have another particular integral

$$A'\frac{0}{0}\left(x^{\alpha+s} + \frac{a_{s+1}}{b_{s+1}}x^{\alpha+s+1} + \frac{a_{s+1}a_{s+2}}{b_{s+1}b_{s+2}}x^{\alpha+s+2} + \&c.\right),$$

in which $A'\frac{0}{0}$ may be replaced by a new arbitrary constant B.

(iii.) 1872. GLAISHER. "On the Relations between the Particular Integrals in CAYLEY's Solution of RICCATI's Equation." 'Philosophical Magazine,' Fourth series, vol. xliii., pp. 433–438.

The relations between U_2, V_2, P_2, Q_2, R_2, S_2 given in art. 17 are obtained. These afford an example of the principle explained in (ii.). See the introduction, p. 763.

(iv.) 1874. BACH. "De l'Integration par les Séries de l'Équation $\frac{d^2y}{dx^2}-\frac{n-1}{x}\frac{dy}{dx}=y$." 'Annales Scientifiques de l'École Normale Supérieure.' Deuxième série, vol. iii., pp. 47–68.

Detailed account, with developments, of (i.) and (iii.). In (iii.) n is written in place of $\frac{1}{q}$ and β in place of $\frac{x^q}{q}$, so that the series are reduced to the forms given in art. 16. If the differential equation is similarly transformed it becomes

$$\frac{d^2u}{d\beta^2}-\frac{n-1}{\beta}\frac{du}{d\beta}-\beta=0.$$

This is the form of the equation adopted by M. BACH, who finally deduces the series in the case of RICCATI'S equation. The form is a very convenient one. See art. 16.

(v.) 1878. GLAISHER. "Example Illustrative of a Point in the Solution of Differential Equations in Series." 'Messenger of Mathematics,' vol. viii., pp. 20–23.

In the well-known expansions quoted in art. 11, viz.

$$\{1-\sqrt{(1-4x)}\}^p=2^px^p\left\{1+px+\frac{p(p+3)}{2!}x^2+\frac{p(p+4)(p+5)}{3!}x^3+\&c.\right\}$$

$$\{1+\sqrt{(1-4x)}\}^p=2^p\left\{1-px+\frac{p(p-3)}{2!}x^2-\frac{p(p-4)(p-5)}{3!}x^3+\&c.\right\}$$

the series are such that if p is an integer, one terminates, and after a certain number of zero terms, recommences and reproduces the other. It follows therefore that the differential equation whose general integral is

$$u=c_1\{1-\sqrt{(1-4x)}\}^p+c_2\{1+\sqrt{(1-4x)}\}^p$$

must afford an example of the principle pointed out in (i.). The differential equation is found to be

$$x(1-4x)\frac{d^2u}{dx^2}+\{(4p-6)x-p+1\}\frac{du}{dx}-p(p-1)u=0,$$

and its integration in series affords the illustration referred to in the title. The note was suggested by art. 11. See art. 15.

(vi.) 1878. —— "Generalised Form of Certain Series." 'Proceedings of the London Mathematical Society,' vol. ix., pp. 197–202.

Theorems deduced from

$$\left(1-x+\frac{n+2}{n+1}\frac{x^2}{2!}+\frac{(n+2)(n+4)}{(n+1)(n+2)}\frac{x^3}{3!}-\&c.\right)e^x$$

$$=\left(1+x+\frac{n+2}{n+1}\frac{x^2}{2!}+\frac{(n+2)(n+4)}{(n+1)(n+2)}\frac{x^3}{3!}+\&c.\right)e^{-x}.$$

See art. 7.

(vii.) 1878. —— "On the Solution of a Differential Equation allied to RICCATI's." 'British Association Report' for 1878 (Dublin), pp. 469, 470.

Proof that the coefficient of h^{i+1} in the expansion of $e^{x\sqrt{(a^2+2h)}}$ satisfies the differential equation

$$\frac{d^2u}{da^2}=a^2u+\frac{i(i+1)}{a^2}u.$$

See arts. 8, 9.

WRITINGS REFERRED TO IN § V.

(viii.) 1813. POISSON. "Mémoire sur les Intégrales Définies." 'Journal de l'École Polytechnique,' vol. ix. (cah. xvi.), pp. 236–239, 241.

It is proved that if

$$y=\int_0^\infty e^{-x^n-\frac{ba^n}{x^n}}dx,$$

then y satisfies the differential equation $\frac{d^2y}{da^2}=n^2ba^{n-2}y$, and it is deduced from this result that the equation is integrable in a finite form when $n=\frac{2}{1\pm 2i}$. See arts. 20, 36.

A relation between two definite integrals is also proved. See art. 26.

(ix.) 1872. GLAISHER. "On the Evaluation in Series of Certain Definite Integrals." 'British Association Report' for 1872 (Brighton), Transactions of the Sections, pp. 15–17.

Investigation of the formula (8) of art. 21 by the process given in arts. 21, 22.

WRITINGS REFERRED TO IN § VI.

(x.) 1839. GASKIN. Senate House Problem.

The solution of the equation

$$\frac{d^2u}{dx^2}+a^2u=\frac{p(p+1)}{x^2}u$$

is given in the forms

$$u=Cx^{-p}\left(\frac{d}{dr}\right)^p\frac{\cos(x\sqrt{r}+a)}{\sqrt{r}},$$

r being put equal to a^2 after the differentiations, and

$$u=Cx^{p+1}\int_{-a}^{a}(r^2-a^2)^p\cos(rx+\alpha)dr.$$

See arts. 37, 39.

(xi.) 1839. HYMERS. 'A Treatise on Differential Equations and on the Calculus of Finite Differences' (1839), pp. 83–85; also, second edition (1858), p. 125.

Solution of Mr. GASKIN's problem in (x.). See art. 37.

(xii.) 1841. ELLIS. "On the Integration of Certain Differential Equations," 'Cambridge Mathematical Journal,' vol. ii., pp. 193–195.

Independent investigation of the [illegible] of Mr. GASKIN's forms in (x.). See art. 37.

(xiii.) 1841. DE MORGAN. 'The Differential and Integral Calculus,' pp. 702–704.

Account of ELLIS's method (see xii.) and of POISSON's determination of the integrable cases of RICCATI's equation (see viii.). See arts. 36, 37.

(xiv.) 1844. BOOLE. "On a General Method in Analysis," 'Philosophical Transactions' for 1844, pp. 251, 252.

This paper contains BOOLE's general symbolic method. The solution of the equation (1) is given in the form

$$u=\frac{1}{x^{i+1}}\left(x^3\frac{d}{dx}\right)^i\frac{c_1e^{ax}+c_2e^{-ax}}{x^{2i-1}}.$$

The general method and this solution are reproduced with only slight changes in BOOLE's 'Differential Equations,' chapter xvii. See art. 34.

(xv.) 1846. LEBESGUE. "Remarques sur l'Équation $y''+\frac{m}{x}y'+ny=0$," 'LIOUVILLE's Journal,' vol. xi., pp. 338, 339.

Solution of this equation in a form involving repeated differentiations with regard to x. See art. 37.

(xvi.) 1848. HARGREAVE. "On the Solution of Linear Differential Equations," 'Philosophical Transactions' for 1848, pp. 34, 35, 45.

The paper contains the general integral of (1) in the forms,

$$u=x^{i+1}(\mathrm{D}^2-a^2)^i\frac{c_1e^{ax}+c_2e^{-ax}}{x},$$

$$u=c_1x^{-p}\int_{-1}^{1}(z^2-1)^{-p-1}e^{axz}dz+c_2x^{p+1}\int_{-1}^{1}(z^2-1)^pe^{axz}dz,$$

and a development of $(\mathrm{D}^2-a^2)^i\frac{e^{ax}}{x}$ in a series. There are also solutions of other allied equations. See arts. 41, 42.

(xvii.) 1856. WILLIAMSON. "On the Solution of Certain Differential Equations." 'Philosophical Magazine,' Fourth series, vol. xi., pp. 364–369.

The general integral of the equation

$$\frac{d^2u}{dx^2}+a^2u=\frac{i(i+1)}{x^2}u$$

is given in the form

$$u=Ax^{-i}\left(\frac{d}{da}a^{-1}\right)^i\cos(ax+\alpha),$$

and the solutions of RICCATI's and several other equations are also obtained. The symbolic expressions are developed by means of the theorem

$$(Da^{-1})^n=a^{-n}D^n-\frac{n(n+1)}{2}a^{-(n+1)}D^{n-1}+\frac{(n-1)n(n+1)(n+2)}{2.4}a^{-(n+2)}D^{n-2}\ldots$$

$$\pm1.3\ldots(2n-1)a^{-(2n-1)}(D-a^{-1}),$$

of which a proof is given. See art. 37.

(xviii.) 1857. DONKIN. "On the Equation of LAPLACE's Functions, &c." 'Philosophical Transactions' for 1857, p. 44.

The integral of the equation in (xvii.) is given in the form

$$x^i\left(D\frac{1}{x}\right)^i(c_1\sin ax+c_2\cos ax).$$

This solution occurs in a note, as an example of the application of the general method of the paper to a particular equation. See art. 37.

(xix.) 1871. GLAISHER. "On RICCATI's Equation." 'Quarterly Journal of Mathematics,' vol. xi. pp. 267–273.

By means of the definite integral (31) of art. 36, the solution of RICCATI's equation is obtained in the forms

$$u=z\left(z^{-2q+1}\frac{d}{dz}\right)^{i+1}(c_1e^{\frac{1}{q}z^q}+c_2e^{-\frac{1}{q}z^q}),\quad\&c.$$

and the formulæ (22) and (23) of art. 31 are proved. See arts. 31, 36.

(xx.) 1872. —— "On a Differential Equation allied to RICCATI'S." 'Quarterly Journal of Mathematics,' vol. xii., pp. 129–137.

The equation is (1), and the definite integral $\int_0^\infty \frac{\cos a\xi}{(x^3+\xi^3)^{\nu+1}}d\xi$ is applied as in art. 29 to obtain the general integral in the form

$$u=x^{\nu+1}\left(\frac{1}{x}\frac{d}{dx}\right)^\nu \frac{c_1e^{ax}+c_2e^{-ax}}{x},$$

and also in BOOLE'S form (29): the results are transformed so as to give the symbolic solution of RICCATI'S equation, which is integrated also by BOOLE'S method. See arts. 26, 30, 33, 34, 35, 38.

(xxi.) 1876. —— "Sur une Propriété de la Fonction $e^{\sqrt{x}}$." 'Nouvelle Correspondance Mathématique,' vol. ii., pp. 240–243, 349–350.

Proof of the theorem

$$2^{2n+1}\left(\frac{d}{dx}\right)^n x^{n+\frac{1}{2}}\left(\frac{d}{dx}\right)^{n+1}e^{\sqrt{x}}=e^{\sqrt{x}}$$

by means of the integral

$$\int_0^\infty e^{-ax^2-\frac{b}{x^2}}dx=\frac{\sqrt{\pi}}{2\sqrt{a}}e^{-2\sqrt{(ab)}}.$$

See art. 36.

(xxii.) 1876. —— "On Certain Identical Differential Equations." 'Proceedings of the London Mathematical Society,' vol. viii., pp. 47–51.

Generalisations of the theorem in (xxi.), as for example

$$\left\{\left(\frac{d}{dx}\right)^n x^{n+\frac{1}{r}}\right\}^{r-1}\left(\frac{d}{dx}\right)^{n+1}\phi(x^{\frac{1}{r}})=\frac{1}{r^{rn+1}}\phi^{(rn+1)}(x^{\frac{1}{r}}),$$

and other similar results. See arts. 33, 36.

(xxiii.) 1879. —— "On a Symbolic Theorem involving Repeated Differentiations." 'Proceedings of the Cambridge Philosophical Society,' vol. iii., pp. 269–271.

The theorem is (50) of art. 40, viz.

$$\left(\frac{d^2}{dx^2}-a^2\right)^n\frac{e^{ax}}{x}=(-)^n2^n.n!\left(\frac{1}{x}\frac{d}{dx}\right)^n\frac{e^{ax}}{x},$$

and the proof is the same as in art. 41.

XIX. The Croonian Lecture.—
Observations on the Locomotor System of Echinodermata.

By George J. Romanes, *M.A., F.R.S., and Professor* J. Cossar Ewart, *M.D.*

Received March 5,—Read March 24, 1881.

[Plates 79–85.]

PART I.—MORPHOLOGY.

§ 1. Ambulacral System.

1. Holothuria.—When a longitudinal incision is made through the perisome of a Holothurian (*Holothuria communis*) there is generally seen escaping, along with the branches of the respiratory tree and genital gland, a long sacculated tube filled with a fluid, and holding in suspension a large quantity of a brick-dust coloured pigment. This tube, which may be one-and-a-half times the length of the entire animal, and from one line to half-an-inch in diameter, is the polian vesicle (Plate 79, fig. 1, *a*). On following it upwards it is found to open freely into a wide circular canal (Plate 79, fig. 1, *b*) a short distance from the termination of the stone canal. From this circular canal five lozenge-shaped sinuses (Plate 79, fig. 1, *c*) project forwards, and from each of these two large oval sinuses (Plate 79, fig. 1, *d*) run forwards parallel with each other, the ten oval sinuses becoming continuous with the hollow stems of the tentacles (Plate 79, fig. 1, *e*). In a Holothurian 8 inches in length, exclusive of the tentacles, the lozenge-shaped sinuses, which may be designated the sinuses of the circular canal, measure a quarter of an inch from above downwards and a little more from side to side. From around the pointed upper ends of the canal sinuses the five longitudinal muscular bands take their origin.

When a solution of Berlin blue is injected into the polian vesicle, the circular canal and its sinuses, the oval sinuses and tentacles, the radial canals, pedicels and ampullæ are rapidly distended; but, unless the pressure be kept up for a considerable time, none of the coloured fluid penetrates into the stone canal, and either the vesicle, ring, or one of the sinuses gives way before it reaches the madreporic plate. If one of the radial canals be divided while the injection is being proceeded with, the coloured fluid at once escapes, and the tension within the polian vesicle, the circular canal, and the tentacles is diminished. If plaster of Paris be substituted for the solution of Berlin blue, a cast is readily obtained of the circular canal and its sinuses, but the plaster does not find its way either into the sinuses of the tentacles or into the radial canals.

When, however, a coloured solution of gelatine is forced into the polian vesicle, the tentacles and their sinuses, the radial canals, ampullæ, and pedicels are filled, as well as the circular canal and its sinuses. Examination of specimens injected with plaster of Paris and gelatine shows the circular canal to be a quarter of an inch in diameter, and to communicate freely both with the polian vesicle and with the lozenge-shaped sinuses springing from it. The specimens injected with a gelatine mass further show that each canal sinus opens into a cæcal tube, which runs forward internal to the sinuses of the tentacles as far as a wide circum-oral space (Plate 79, fig. 2, *a*). This space communicates by well-defined apertures (Plate 79, fig. 3, *b*), with that portion of the body cavity which lies between the sinuses and the œsophagus, and which is reached through the circular apertures between the sinuses of the circular canal (Plate 79, fig. 1, *f*).

Each canal sinus has three other apertures in its walls. It opens by a small round aperture into a radial canal. The radial canal, together with the longitudinal muscular band, runs up between two of the sinuses of the tentacles to reach the inner surface of the body wall, and gives off lateral branches which project outwards as pedicels and inwards at each side of the longitudinal muscle as long-pointed ampullæ (Plate 79, fig. 1, *g*). The two other apertures are in the form of minute slits, one at each side of the orifice of the radial canal, which lead into the adjacent tentacle-sinuses. Each of these tentacle-sinuses measures three-quarters of an inch in length, and a little over a quarter of an inch in diameter. When the tentacle into which the sinus opens is protruded, there is no constriction between the sinus and the tentacle; but when the tentacle is retracted, there is a well-marked annular constriction (Plate 79, fig. 1, *h*) at the junction of the sinus with the tentacle, which may aid in preventing the fluid that is driven into the sinus during retraction from again returning into and at once protruding the tentacle. If considerable pressure be applied to the polian vesicle when the tentacles are in a retracted state, this constriction disappears and the tentacles are distended, though not protruded. If the retractor muscles be divided while the polian vesicle is compressed, the tentacles become engorged and project forwards. On the other hand, when the pressure is removed from the vesicle and the divided retractors pulled backwards, the tentacles are approximated and, along with the group of sinuses and the anterior portion of the perisome, dragged backwards towards the centre of the body cavity. The eversion of the perisome and the protrusion of the tentacles are brought about chiefly by the shortening of the longitudinal muscular bands and the contraction of the polian vesicle; but the circular fibres of the body wall also assist by contracting immediately behind the group of sinuses, so as to act on them by direct pressure, and also indirectly by forcing the perivisceral fluid against them. The fully distended position of the tentacles represented in figure 3 is only reached when the fluid of the body cavity has been forced into the circum-oral space (Plate 79, fig. 2, *a*).

The amount of the body cavity fluid is constantly changing. At the entrance to the

cloacal chamber a circular valve is seen alternately dilating and contracting, except when the aboral end of the Holothurian is forcibly retracted. When this valve dilates it lies in close contact with the walls of the cloaca (Plate 79, fig. 4) and allows water to enter for the respiratory tree. It remains open for a few seconds and then contracts so as to project beyond the aperture (Plate 79, fig. 5); when it begins to retract and dilate, water escapes from the cloaca. This alternate opening and closing takes place rhythmically, at a rate usually of six revolutions per minute. At the end of every seventh or eighth revolution the valve projects further than usual, and, while it is slowly dilating, a large stream of clear water is ejected. The escape of this stream occupies from 15 to 20 seconds. Occasionally along with the stream a quantity of sand and the remains of the food particles are carried out from the cloaca. When the tentacles are being protruded the rhythmic action of the valve goes on as before, but more water is taken in than escapes from the cloaca: on the other hand, retraction of the tentacles is preceded by an escape of a large stream of water, and while retraction is proceeding more water escapes than when the Holothurian is at rest with its tentacles projecting.

2. Echinus.—In *Echinus* (*E. sphæra* and *lividus*) two tubes spring from the under surface of the madreporic plate. The one (Plate 80, fig. 10, *a*) is dilated at its origin so as to include the greater portion of the plate (Plate 80, fig. 10, *b*), and ends in the so-called "heart" (Plate 80, fig. 10, *d*). The other (Plate 80, fig. 10, *c*) is small, deeply pigmented, and runs along a groove in the heart to open into a circular canal at the base of the lantern. From the under aspect of this circular canal the five radial ambulacral vessels (Plate 80, fig. 10, *i*) take their origin, and, after passing under the rotulæ and over the interalveolar muscles, they run along the inner surface of the ambulacral plates. The first series of pedicels (Plate 80, fig. 10, *j*) projects through the oral floor midway between the oral aperture and the margin of the shell. Their respective ampullæ (Plate 80, fig. 10, *k*) are long delicate tapering tubes which project upwards and outwards between the radial canals and the alveoli. The next four or five pairs of pedicels also pierce the oral floor. Their ampullæ are small rounded sacs, whereas the ampullæ of the first series of feet external to the auricles are slightly flattened and sometimes deeply constricted, whilst those beyond are in the form of flattened sacs lying at right angles to the radial canals (Plate 80, fig. 13, *a*).

Immediately within the oral margin of the shell, and alternating with the inner row of pedicels, are the five pairs of "tree-like organs." There is no evidence of the existence of these structures within the shell until a membrane (Plate 80, fig. 10, *m*), which extends from the apex of each tooth to the oral margin of the interambulacral plates and sides of the alveoli, is divided. If instead of dividing this membrane a fine glass canula be forced through it and a solution of Berlin blue introduced into the space between the membrane and the alveoli of the lantern, the fluid slowly diffuses upwards into, and greatly distends, the vesicles around the apices of the teeth (Plate 80, fig. 10, *n*). The fluid reaches these vesicles partly by passing directly upwards

external to the alveoli, and partly by passing into the cavities of the alveoli and ascending through the circular sinus. When a solution of coloured gelatine, or what is better, plaster of Paris, is injected into the space above the tentacles, or into the vesicle around the soft apex of one of the teeth, a cast is readily obtained of the circular sinus and of the spaces communicating with it. A vertical section of the lantern of an urchin thus injected shows the wedge-shaped circular sinus (Plate 80, fig. 10, *e*) lying between the radius and rotula, communicating above with the vesicle around the tip of the tooth (Plate 80, fig. 10, *n*) and below with the cavity of the alveolus (Plate 80, fig. 10, *p*), through which the tooth passes, and by means of the latter cavity communicating indirectly with the large space lying above the "tree-like organs."

3. Spatangus.—In *Spatangus* (*S. purpureus*) the ambulacral circum-oral canal has no polian vesicles or sinuses developed in connexion with it. The ampullæ immediately around the oral aperture are tubular, and often measure a quarter of an inch in length. Those beyond on the under surface and round the equator are also tubular, but they are small and few in number. This is true of all the ampullæ of the anterior radial canal. The ampullæ which project inwards from the dorsal portions of the four remaining radial canals are, as in *Echinus*, transversely flattened sacs. Some of the pedicels have suckers, others are conical and devoid of suckers, while others again are flattened at their tips, or flattened and split up into segments.

4. Solaster.—When one of the arms of *Solaster papposa* is divided transversely an inch from the disc, and a coloured solution is introduced into the proximal portion of the radial canal, the ampullæ and pedicels at the base of the arm injected are at once distended. The solution next penetrates the circular canal, polian vesicles, ampullæ, and pedicels of the other arms; but unless a considerable pressure be continued for some time, none of the solution enters the madreporic canal. In one specimen, when the injection had been continued for five hours with the pressure bottle raised two feet above the level of the Star-fish, the solution had ascended two-thirds of the entire length of the stone canal, and two hours later it began to diffuse slowly through the madreporic plate. When a thin slice was removed from the surface of the plate, the solution was observed escaping from a small circumscribed area (Plate 80, fig. 12, *a*) situated between the centre and margin of the plate—an area corresponding in size and position with the termination of the stone canal on the inner surface.

Starting from the inner aspect of the madreporic plate the stone canal gradually increases in diameter, and passes obliquely over the accompanying sinus till it finally hooks round the sinus to open into the circular canal (Plate 80, fig. 11, *a*). Springing from this canal, which occupies a sinuous groove on the dorsal aspect of the inner ambulacral ossicles, and opposite each interradial space (with the exception of the space occupied by the stone canal) is a polian vesicle (Plate 80, fig. 11, *c*). Each vesicle consists of a tubular stem measuring from two lines to a quarter of an inch in length, and of a dilated portion which may be exceedingly small (Plate 80, fig. 11, *i*), or takes

the form of a large oval sac (Plate 80, fig. 11, *d*). The size and form of the vesicles are largely determined by the amount of fluid in the pedicels, the vesicles diminishing when the feet are protruded, and enlarging when they are retracted.

The first series of ampullæ (Plate 80, fig. 11, *e*) are small semi-lunar sacs, which lie in close contact with each other opposite the origin of the polian vesicles, and between the ambulacral canal and the circular vessel. The other ampullæ (Plate 80, fig. 11, *f*) are spherical in form. In none of the injected specimens was there any evidence of a communication between the ambulacral vessels and the body cavity, or between the ambulacral and the blood (neural) vessels. There was, however, abundant evidence of communication between the latter and the exterior. When a canula was introduced into the outer end of the sinus, a coloured watery solution could be easily forced through the sinus into the circular blood-vessel (Plate 80, fig. 11, *h*), and from the circular vessel into the radial blood-vessels. But when the canula was introduced into the proximal end of the sinus, the solution rapidly rushed along the sinus and escaped freely through the madreporic plate—proving that the blood-vessels of *Solaster* communicate far more freely with the exterior than do the water vessels.

5. Uraster and Astropecten.—The ambulacral system of the common Star-fish (*Uraster rubens*) differs from that of the Sun-star (*Solaster*) only in having no polian vesicles. *Astropecten* (*A. aurantiaca*), on the other hand, has polian vesicles; but in it the pedicels have departed from the typical form. In *Holothuria* and *Echinus* the feet terminate in well-marked sucking discs (Plate 79, fig. 6), which have their margins frequently strengthened by a deposit of calcareous matter. All the pedicels of *Solaster* and *Uraster*, with the exception of a few at the tip of each arm, are also provided with suckers. Those at the tips of the arms are long and pointed, and when the Star-fish is moving they project forwards in the direction of advance, and appear to act the part of tentacles. In *Astropecten*, however, the feet are short and conical, and instead of ending in suckers they terminate in rounded points (Plate 79, fig. 7). But although suckers are absent, *Astropecten* is able slowly to ascend a vertical surface. We have repeatedly observed slight inversions of one side of the pedicels near their pointed tips when this Star-fish slowly ascended from the bottom of a glass aquarium; hence we are inclined to believe that *Astropecten* has the power of converting a portion of the side of its pedicel into an imperfect sucker (Plate 79, fig. 8, *a*).

6. Ophiura.—In the Brittle and Sand-stars, the ambulacral feet are morphologically similar to those of *Astropecten*, though shorter and more slender. Those near the disc further differ from the pedicels of *Astropecten* in being more pointed (Plate 79, fig. 9). The pedicels beyond the disc gradually diminish in size, and at the ends of the arms they are scarcely visible. All the feet are devoid of suckers, and no attempt is ever made to form even a temporary imperfect sucker by slightly inverting a portion of the side of the foot.

§ II. General Homologies.

The madreporic plate of a Holothurian is a pale, straw-coloured, hollow, conical, calcareous mass, lying on or near the circular canal. It may, however, be broken up into several portions; from each portion a canal originates, and the several canals generally unite to form a single stone canal. The stone canal, though sometimes straight, and hanging from the circular canal into the body cavity, is generally folded on itself, and in contact throughout the greater portion of its length with the circular canal. A small sinus, which sometimes exists around the stone canal, may correspond to the large sinus which lies in contact with the stone canal of *Solaster*.

In *Echinus* the madreporic plate is a modified genital plate, the stone canal is a delicate pigmented fibrous tube, lined with cells provided with long vibratile cilia. This tube springs from a limited area of the plate, and runs along the "heart" to open into the circular canal at the base of the lantern. The sinus, springing from the larger portion of the plate, contracts into a narrow tube, and then dilates and has developed in its walls a lobulated glandular-looking mass, which may act the part of an excretory organ in connexion with the vascular system.

In *Solaster*, *Uraster*, and *Astropecten*, the madreporic plate, though still placed on the dorsal aspect, has been removed from the genital and ocular plates by the appearance of antambulacral ossicles. A small calcareous stone canal and a wide membraneous sinus arise from the inner surface of the madreporic plate. The stone canal opens into a narrow circular canal; the sinus runs along under the stone canal, and, after diminishing considerably in size, opens into a circular blood-vessel. From this circular blood-vessel the radial (neural) vessels take their origin. A small glandular-looking mass, which lies in the floor of the sinus, may correspond to the glandular portion of the "heart" of *Echinus*. In *Ophiura* the madreporic canal springs from one of the interradial oral plates. Before opening into the circular canal it dilates into a vesicle.

The wide circular canal of the Holothurian corresponds to the circular canal at the base of the lantern of *Echinus*, and to the narrow canals of *Solaster*, *Uraster*, *Astropecten*, and *Ophiura*.

The long single polian vesicle of the Holothurian corresponds to the polian vesicles of *Solaster* and *Astropecten*, and to those of *Ophiura* when they are present. The lozenge-shaped sinuses of *Holothuria* have apparently nothing homologous to them in either the Sea-Urchins or the Star-fish; but the large oval sinuses of *Holothuria* may correspond to the sinuses lying over the rudimentary tentacles of *Echinus*. In *Holothuria* the radial canals take their origin from the sinuses of the circular canal; in all the other types mentioned the radial canals arise directly from the circular canal.

In *Holothuria* the ampullæ are long and pointed; in the Star-fish they are spherical; in *Echinus* the first series are conical, while the others within the auricles are rounded and those without the auricles are large transversely flattened sacs. In *Ophiura* the ampullæ are absent.

In *Holothuria* and *Echinus* the pedicels are provided with sucking discs, which are sometimes strengthened by a calcareous skeleton. In both, but especially in *Echinus*, they can be projected to a considerable distance beyond the surface. In *Spatangus* the feet are comparatively short, and although some have suckers, many are without them and end in simple rounded points, while others are either simply flattened at their apex, or flattened and split up into leaf-like segments. In *Solaster* and *Uraster* the feet are long, and terminate in large sucking discs, with the exception of a few at the end of each arm, which are pointed, and act as feelers. In *Astropecten* the feet are conical, devoid of suckers, and can only be projected about a quarter of an inch beyond the surface. In *Ophiura* the feet are even more pointed and shorter than those of *Astropecten*. Under the disc at the bases of the arms they are nearly as long as in *Astropecten*; but they gradually diminish in size from within outwards, until near the tips of the arms it is almost impossible to recognise them.

§ III. Nervous System of Echinus.

The internal nervous system of *Echinus* consists of five radial trunks, which may be traced from the ocular plates along the ambulacral areas external to the radial canals to the oral floor, where they bifurcate and unite with each other, so as to form a pentagonal nerve-ring. This ring lies between the œsophagus and the tips of the teeth which project from the lantern. Small branches leave the ring and supply the œsophagus, and lateral branches arise from the several trunks to escape with the pedicels through the apertures of the pore plates. Each trunk lies in a sinus (Plate 80, fig. 13, *c*) situated between the lining membrane of the shell (Plate 80, fig. 13, *d*) and the ambulacral radial canal (Plate 80, fig. 13, *e*); the lateral branches which accompany the first series of pedicels through the oral floor are large and deeply pigmented; the other branches within the auricles are small; those external to the auricles gradually increase in size until the equator is reached, and from the equator to the ocular plates they again diminish. At the equator the trunk is wider than at either pole, and it is often partially divided for some distance at each side of the equator by a deep longitudinal fissure. When the nerve trunk, after being stained with chloride of gold or with osmic acid, is removed from its sinus, it is seen to be enveloped by a thin fibrous sheath. This sheath contains numerous large pigment cells, and has scattered over it irregular masses of protoplasm which have been deposited from the fluid of the neural sinus.

When the sheath is removed the trunk is seen to consist of delicate fibres and of fusiform cells (Plate 80, fig. 14); the cells consist of a nucleus and a thin layer of protoplasm, which projects at each end and terminates in a nerve-fibre.

The lateral branches of the trunk escape along with, and are partly distributed to, the pedicels; the remainder breaks up into delicate filaments which radiate from the base of the pedicel under the surface epithelium (Plate 80, fig. 13, *l*). When one of

the large branches already referred to as escaping with the inner row of pedicels is traced through the oral floor after sending a branch to the foot, it breaks up into delicate fibres, some of which run towards the bases of the adjacent spines and pedicellariæ, while others run inwards a short distance towards the oral aperture.

Either in connexion with, or anatomically independent of these filaments from the lateral branches of the nerve trunks, there is an external plexus lying almost immediately under the surface epithelium and extending from the shell to the spines and pedicellariæ. The fibres (Plate 80, fig. 15) of this plexus closely resemble those of the lateral branches of the trunk; but generally they are smaller in size and have a distinct connexion with nerve cells. The cells consist of an oval nucleus and of a layer of protoplasm, which is generally seen to project in two, or sometimes in three, directions—the several processes often uniting with similar processes from adjacent cells so as to form a fibro-cellular chain or network.

In preparations from portions of Echini treated with both chloride of gold and osmic acid, we have succeeded in tracing the plexus over the surface of the shell between the spines and pedicellariæ; and from the surface of the shell to the capsular muscles at the bases of the spines (Plate 80, fig. 16). Further, we repeatedly observed delicate fibres passing beyond the muscles, apparently to end under the epithelium over the surface of the spines (Plate 80, fig. 13, *l'*).

In the case of the pedicellariæ, the plexus on reaching the stem runs along between the calcareous axis and the surface epithelium, to reach and extend over and between the muscular and connective tissue-fibres between the calcareous axis and the bases of the mandibles (Plate 80, fig. 13, *l''*, and fig. 18). The plexus, now in the form of exceedingly delicate fibres connecting small bipolar cells, reaches the special muscles of the mandibles. In several preparations, delicate fibres appeared to extend towards the sensitive epithelial pad (Plate 80, fig. 13, *s*) situated on the inner surface of each mandible, a short distance from the apex. Although this plexus is especially related to the muscular fibres—lying over and dipping in between them—it is also related to the surface epithelium, and delicate fibres often extend from it to end under or between the epithelial cells.

PART II.—PHYSIOLOGY.

§ I. Natural Movements.

It is desirable to begin this account of the physiology of the locomotor system, with a somewhat full account of the natural movements exhibited by the various species of Echinoderms which we have had the opportunity of observing. This is desirable, not only because it is necessary to study the natural movements before we can be in a position to appreciate the results of the following experiments, but also because these natural movements form in themselves a study of considerable interest.

1. Star-fishes.—(A) Taking the common Star-fish (*Uraster rubens*) as our start-

ing point, it is needless to dwell upon the well-known mechanism of the ambulacral system. The rate of crawling upon a flat horizontal surface is 2 inches per minute. The animal usually crawls in a determinate direction, and, while crawling, the ambulacral feet at the end of each ray are protruded forwards as feelers; this is particularly the case with the terminal feet on the ray, or rays, facing the direction of advance. When in the course of their advance these tentacular feet happen to come into contact with a solid body, the animal may either continue its direction of advance unchanged, or may deflect that direction towards the solid body. Thus, for instance, if, while the Star-fish is advancing along the floor of a tank, the tentacular feet at the end of one of its rays happen to touch a perpendicular side of the tank, the animal may either at once proceed to ascend this perpendicular side, or it may continue to progress along the floor, feeling the perpendicular side with the ends of its rays perhaps the whole way round the tank, and yet not choosing,* as it were, to ascend. What it is that determines the animal in some cases to ascend, and in other cases not, we were unable to ascertain.

When a Star-fish ascends the perpendicular side of a tank or bell-jar till it reaches the surface of the water, it very frequently performs a number of peculiar movements, which we may call acrobatic (see Plate 81, fig. 19). On reaching the surface of the water, the animal does not wish to leave its native element, and neither does it wish again to descend into the levels from which it has just ascended. It therefore begins to crawl to one side or the other, and while crawling it every now and then throws back its uppermost ray, or rays, to feel for any solid support that may happen to be within reach. The distance to which the rays may thus be thrown back is remarkable; for the animal may hold on with its two lower rays alone, or even with the end of a single ray, and throw back the whole of the other rays with the central disc into a

* It may be as well to explain that in using such words as these, we do not, in the present paper, attach to them any psychological signification; they are used as merely metaphorical terms which serve most briefly, and therefore most conveniently, to express the resultants of those systems of physiological stimuli, the composing members of which we were not able to observe. When one Star-fish appears to choose to ascend the side of a tank, while another Star-fish, under apparently precisely similar circumstances as to stimulation, seems to prefer walking along the floor, we can only suppose that the circumstances of stimulation, although apparently similar, are not really so, and therefore that the difference in the result is due to some difference in the stimulation. Of course it may be objected to this that the same remark applies to cases in which the psychological element unquestionably enters—choice on its physiological side being merely the resultant of some unobservable system of stimuli. But without here entering on the whole question of the relation between body and mind, it is enough to point out that the only evidence we can have of a physiological determination presenting a psychological side, is by observing that the organism which exhibits the determination is capable of altering it on future occasions, if the determination first made is found by individual experience to be injurious. In other words, the power of learning by individual experience is the only unequivocal evidence we can possess of the presence, in any animal, of a psychological element; and as we have observed no such evidence in the case of any of the Echinoderms, we desire it to be understood that we consider all their movements to be of the so-called "reflex" kind.

horizontal position—the ambulacral surface of the rays which are thrown back being then of course turned up, so as to face the surface of the water. If the rays succeed in finding a solid body, they will perhaps—though not necessarily—fasten upon it, and when their hold is secure, the rays which hitherto held the animal to the side of the tank relax their suckers, so that the Star-fish swings from its old to its new surface of attachment. The activity and coordination which the rays manifest in executing these various acrobatic movements is surprising, and give to the animal an almost intelligent appearance.

If a Star-fish is turned over on its dorsal surface upon the flat floor of a tank, it almost immediately begins to right itself. Its method of doing so (see Plate 81, fig. 20) is to twist round the tip of one or more of its rays (*a*), until the ambulacral feet there situated are able to get a firm hold of the floor of the tank; then by a successive and similar action of the ambulacral feet further back in the series, the whole end of the ray is twisted round (*b*), so that the ambulacral surface of the end is applied flat against the flat surface of the tank (*c*). The manœuvre continuing, the semi-turn or spiral travels progressively all the way down the ray. Usually two or three adjacent rays perform this manœuvre simultaneously; but if—as is sometimes the case—two opposite rays begin to do so at first, one of them soon ceases to continue the manœuvre, and one or both of the rays adjacent to the other takes it up instead. The spirals of all these rays being turned in the same direction (see *a*, *b*, *c*), the result is, when they have proceeded sufficiently far down the rays, to drag over the disc and the remaining rays (*d*, *e*), which abandon their hold of the bottom of the tank, so as not to offer any resistance to the lifting action of the other rays; the animal, therefore, turns a complete somersault—the disc and inactive rays being thrown over the active ones with considerable rapidity. The whole movement—from the first twisting round of the tips of the active rays to the final turning over of the whole animal—does not usually occupy more than about half a minute. It will be seen that this whole movement implies no small amount of co-ordination, and it is therefore of interest to consider it in this connexion. As a general rule, the rays are from the first co-ordinated to effect the righting movement in the direction in which it is finally to take place—the rays which are to be the active ones alone twisting over, and so twisting that all their spirals turn in the same direction. This, however, although usually, is by no means invariably the case; for at the commencement of the righting movement different rays may act in antagonistic ways—twisting their spirals in opposite directions, and doubling their ends under, without reference to the direction in which the somersault is eventually to be turned. But in all cases a definite plan, so to speak, is very soon made—the opposition rays, as previously stated, leaving go their hold, the antagonistic spirals of adjacent rays being unwound or reversed, while any antagonistic doublings are straightened out; so that the whole righting movement in fresh specimens never, at the most, occupies more than a minute.

(B.) Sun-stars (*Solaster*).—All the remarks which have been made on the natural

movements of the common Star-fish, are equally applicable to the Sun-stars. When placed on the dorsal surface, however, their righting movements are conducted on a slightly different plan. Owing to the disc being here so large in proportion to the length of the rays, it would be useless in the latter to endeavour to turn over the former by twisting themselves into spirals. They therefore adopt a device which in the common Star-fish is sometimes made accessory to that of twisting the rays, and which is also shown in Plate 81, fig. 20;* they double under the ends of a number of adjacent rays, laying hold of the floor of the tank with their ambulacral feet as the doubling progressively advances up the length of the ray. When this doubling has advanced up a considerable length of a number of adjacent rays, the ambulacral feet upon these rays obtain a sufficient purchase to drag over the whole of the large disc in a manner otherwise similar to that which has just been described in the case of the common Star-fish.

(C.) Astropecten aurantiacus.—The ordinary locomotor movements of this species are highly peculiar. The form of the animal very much resembles that of the common Star-fish, although its disc is proportionally larger, and the whole animal smaller. Its ambulacral feet are pointed tubes, rather less than a quarter of an inch long, and, as before stated, unprovided with any sucker at the tip. When the animal is not walking, these feet are nevertheless in a constant state of movement, and their movements are then of a peculiar writhing, almost vermiform character—twisting about in various directions, and frequently coiling round each other. When fully protruded, however, they are perfectly straight and stiff. Their protrusion—whether complete or partial—takes place with great suddenness, and at all times—whether the animal is stationary or not—a number of feet are being protruded, while a number of others are as continually being retracted. The feet usually remain extended for a considerable though indeterminate time (quarter to half a minute), and then very suddenly again collapse. These movements of protrusion and retraction are so sudden that the eye is unable to follow them, and as they are always taking place over a large number of feet at the same time, the appearance presented by the whole series is that of a continual flick-flacking. The erection of the feet takes place obliquely from the median line of the ray, and the collapse takes place laterally—the feet therefore falling over upon the sides of the ray. The animals, as previously observed, can crawl up perpendicular surfaces in the manner previously described; but, owing probably to the absence of any differentiated structures in the form of sucking discs, they soon tire—never succeeding in crawling more than a few inches up the side of a tank before they drop off.

The ordinary locomotor movements of this species are, as we have said, highly

* This figure has been drawn so as to show both these devices. Very often the common Star-fish does not double under the ends of the rays at all, as represented in the figure; but depends entirely on the spiral rotation of the rays for the execution of its righting manœuvre—the dorsal aspect of the active rays being therefore not raised from the floor of the tank as represented.

peculiar, and they may best be studied by taking the animal out of the water, placing it upon a dry flat surface, and watching the movements of its feet by placing the eye on a level with them. It may then be observed that the mode of locomotion is as follows:—The animal points all the feet of all the rays in the direction of advance, and then simultaneously distends them with fluid; they thus become so many pillars of support, which raise the animal as high above the flat surface as their own perpendicular length. The fluid is then suddenly withdrawn, and the Star-fish falls forward flat with a jerk. This manœuvre being again and again repeated at intervals of about a quarter of a minute, the animal progresses in a uniform direction at the rate of about an inch per minute. It is particularly noteworthy that, in this mode of progression, all the feet of all the rays are co-ordinated in their action for determining one definite direction of advance—those in the ray facing that direction acting forwards, or centrifugally, those in the hinder rays backwards, or centripetally, and those in the lateral rays sideways.

When the animal is walking along a flat horizontal surface in water, its mode of progression appears to be the same as it is on a dry surface, only the motion of the feet is now so rapid that there is a considerable difficulty in following it with the eye. It appears, however, as if the feet, besides being used as walking-poles in the manner just described, are also used to sweep backwards along the floor of the tank, and so to assist in propelling the animal forwards after the manner of cilia. Therefore, while walking in water, this Star-fish is kept stilt-high above the surface on which it is walking, by some of its feet, while others of its feet are engaged in these sweeping movements.

The result of all these movements is to produce a kind of locomotion which would seem more suited to a Centipede than to a Star-fish, and the suggestion that it is so is borne in upon the mind all the more forcibly by the surprising rate at which the animal is thus enabled to move. For while an ordinary Star-fish only crawls at the rate of 2 or 3 inches per minute, *Astropecten* can crawl, or perhaps more correctly run, at the rate of between 1 and 2 feet per minute.

When placed upon their backs, the righting movement of these Star-fish are performed by raising the disc from the floor of the tank, till the animal rests only on the tips of its five rays (Plate 81, fig. 21). Two rays—for instance, 4 and 5—are then bent under the disc, while 2 and 3 are raised on a level with the disc. The disc becoming tilted in the direction of 4 and 5, 2 and 3 are now thrown over the disc, and assist by their weight in revolving the whole system upon an axis situated at about the level A, A. This mode of executing the righting manœuvre is somewhat similar to that which occurs in the Sun-stars, only in this case the disc is raised entirely from the floor of the tank, and the whole movement is performed without any aid from the ambulacral feet; the latter, however, are kept in active motion during the whole of the righting movement. Sometimes only one arm, instead of two, is used as the fulcrum over which the disc and the other arms are thrown. In all cases

the righting is effected with much more energy than in the case of any of the species previously mentioned.

(D.) Brittle-stars. (*Ophiuridæ*).—In these Star-fish the ambulacral feet have been reduced to rudiments, which, however, are exceedingly active—their mode of protusion and retraction being precisely similar to that which has just been described in the case of *Astropecten*. Indeed, their activity is even greater in the case of the Brittle-stars; but as they are very short, and not provided with suckers, it does not appear that they are of any use in assisting locomotion. The Brittle-stars, however, are much the most actively locomotive of all the Star-fish; and the reason is that, having discarded the method of crawling by the ambulacral system, which is common to nearly all the other Echinoderms, they have adopted instead a completely new, and a much more effectual method. As the family name of the group implies, the muscular system of the rays is very perfectly developed, enabling these long and snake-like appendages to perform with energy and quickness a great variety of snake-like writhings. As the movement of all the arms is co-ordinated, the animal is able by these writhings to shuffle itself along flat horizontal surfaces at a considerable speed. But when it desires to move still more rapidly, it adopts another plan. If the animal is advancing in the direction of the arrow (Plate 82, fig. 22), one of its rays, 1, is pointed straight in that direction; the two adjacent rays, 2 and 3, are thrown forwards as far as possible, and then, by a strong contraction downwards upon the floor of the tank, these two rays partly elevate the disc, and, while keeping the disc so elevated, throw themselves violently backwards into the form of crescents, as represented in 2′ and 3′. The result of this movement is to propel the animal forwards —ray 1 being pushed into the position 1′, while rays 4 and 5 are dragged along into the positions 4′ and 5′. As soon as the rays 2 and 3 have assumed the position 2′ and 3′, they are again, without an instant's delay, protruded straight, to be again as instantly thrown into the form of the curves 2′ and 3′. Thus the animal advances by a series of leaps or bounds, which vary between $1\frac{1}{2}$ and 2 inches in length, and which follow one another with so much rapidity, that a lively Brittle-star can easily travel at the rate of 6 feet per minute. While thus travelling, the ray, 1, is usually kept straight pointed and partly uplifted—doubtless in order to act as a feeler; but sometimes the animal varies its method of progression, so as to use two pairs of arms for the propelling movements, and in this case the remaining arm is, of course, dragged behind, and so rendered useless as a feeler. The Star-fish is able to use any pair, or pairs, of its arms as propellers indifferently, and in all cases it so uses them by resting their outer, or distal, thirds upon the tank floor, and at each leap raising their remaining two-thirds, together with the anterior part of the disc, off the floor; at the end of each leap, however, the whole animal (except, perhaps, the elevated feeler-ray) lies flat upon the floor.

Brittle-stars, when placed upon their backs, adopt the same method of righting themselves as has already been described in the case of *Astropecten*. They are,

however, even more energetic in executing their righting movements—raising their discs high above the tank floor upon their long arms, and completing their movements in a few seconds. So vigorous are these Star-fish, that they are able to execute this manœuvre even upon a dry table, although the weight to be overcome is so much greater in air than in water.

It may be added that these Star-fish are not able to ascend perpendicular surfaces, owing to the rudimentary condition of their ambulacral apparatus.

2. Echini.—In striking contrast to the rapid locomotion of the Echinoderms last considered, stands the slow locomotion of the *Echinus*, which along a horizontal surface takes place at the rate of only 6 inches per minute, and up a perpendicular surface at the rate of 1 inch in 4 minutes. Looking to the absurdness of this rate of locomotion, it must strike us as a curious fact that there is, perhaps, no animal which can properly be said to approach the *Echinus* in respect of the number and elaboration of special mechanisms subservient to the function of locomotion. Careful observation has satisfied us that these special mechanisms are four in number, and each of these displays an immense amount of elaboration. We may best consider these four mechanisms by taking them separately.

(A.) *Ambulacral feet, or pedicels.*—This system is both structurally and functionally closely similar to the homologous and analogous system in Star-fishes. In the *Echinus*, however, it is of more use than in the Star-fish as a system of anchors and feelers. The form of the *Echinus* being globular, while that of the Star-fish is flat, it follows that the animal is more exposed to the displacing influence of currents, because offering a larger surface for their action. Consequently, a need arises for a more secure system of attaching the animal to the surfaces over which it may be crawling, and this need is supplied by the ambulacral feet acting more the part of anchors than they do in the Star-fish. Thus it is that in forcibly removing an *Echinus* from whatever surface it may be adhering to, a much greater resistance is encountered than one finds in the case of Star-fish, and—especially if a little time is given to the animal after a first alarm to establish a firmer hold—the suckers stick so tightly that a certain number allow themselves to be torn from the organism rather than leave go their attachments—these suckers being therefore left behind, fastened upon the surface to which they were adhering. Under similar circumstances a Star-fish will never thus leave its suckers behind. Indeed, a Star-fish does not seem to fear abandoning itself to the mercy of currents; for, as we shall subsequently see, a very small amount of provocation will induce it to abandon its hold of a perpendicular surface spontaneously, in order to effect its escape by falling through the water. An *Echinus*, on the other hand, always seems, as it were, nervously anxious about its anchorage—in all its movements its first concern appearing to be to have its steadiness amply secured by a sufficient number of suckers, and this even in the perfectly still water of a tank.

The other function of the pedicels which is peculiar to the *Echinus*, viz., that of feelers, also no doubt arises from the shape of the animal; for while in the Star-fish

the pedicels are confined to the ventral surface of a flat-shaped organism, in the *Echinus* they are protruded from all sides of a globe. That they are habitually used as feelers is evident from watching their movements. For instance, when an *Echinus* is crawling along a flat horizontal surface, the rows of pedicels facing the direction of advance are more strongly protruded than those of the other rows; although none of the pedicels from some distance below the equator are in use for walking, in the rows mentioned they are extended to their fullest length, in order to feel for any object which the animal may possibly be approaching. On the other rows only a single pedicel here and there is thus fully extended; such, however, no doubt also act as feelers, to warn the *Echinus* of the approach of any object from behind or from the sides. When a perpendicular surface is reached, the animal may either ascend it or crawl along for an indefinite distance, feeling it all the way with its pedicels. It may here be added that when an *Echinus* starts walking, it generally keeps pretty persistently in one direction of advance. If it be partly rotated by the hand, or other external means, it does not continue in the same direction, but continues its own movements as before; so that, for instance, if it has been turned half round, it will proceed in a direction opposite to that in which it had been proceeding before its rotation. When fresh specimens are at rest, a certain small percentage of feet are used as anchors. The others are strongly protruded on all sides as feelers; but in specimens not quite fresh, nearly all the feet not in use as anchors are retracted, with only one here and there protruded as a feeler.

When an *Echinus* is inverted upon its ab-oral pole, its shape renders execution of the righting manœuvre a much more difficult matter than is the case in the analogous position of a Star-fish; for while a Star-fish is provided with flat, flexible, and muscular rays, composing a small and light mass in relation to the motive power, an *Echinus* is a rigid, non-muscular, and globular mass, whose only motive power available for conducting the evolution is that which is supplied by relatively feeble pedicels. It is therefore scarcely surprising that unless the specimens chosen for these observations are perfectly fresh, they are unable to right themselves at all; they remain permanently inverted till they die. But if the specimens are fresh, they sooner or later invariably succeed in righting themselves, and their method of doing so is always the same. Two, or perhaps three, adjacent rows of pedicels are selected out of the five, as the rows which are to accomplish the task (Plate 82, fig. 23). As many feet upon the rows as can reach the floor of the tank are protruded downwards and fastened firmly upon the floor; their combined action serves to tilt the globe slightly over in this direction—the anchoring feet on the other, or opposite, rows meanwhile releasing their hold of the tank floor to admit of this tilting (Plate 82, fig. 24). The effect of the tilting is to allow the next feet in the active ambulacral rows to touch the floor of the tank, and when they have established their hold, they assist in increasing the tilt; then the next feet in the series lay hold, and so on, till the globe slowly but steadily rises upon its equator (Plate 83, fig. 25). The difficulty of raising such a heavy mass

into this position by means of the slender motive power available can be at once appreciated on witnessing the performance, so that one is surprised, notwithstanding the co-ordination displayed by all the suckers, that they are able to accomplish the work assigned to them. That the process is in truth a very laborious one is manifest, not only from the extreme slowness with which it takes place, but also because in the case of not perfectly strong specimens complete failure may attend the efforts to reach the position of resting on the equator—the *Echinus* after rearing up a certain height, becoming exhausted and again falling back upon its ab-oral pole. Moreover in some cases it is interesting to observe that when the equator position has been reached with difficulty, the *Echinus*, as it were, gives itself a breathing-space before beginning the movement of descent—drawing in all its pedicels save those which hold it securely in the position to which it has attained, and remaining in a state of absolute quiescence for a prolonged time. It then suddenly begins to protrude all its feet again, and to continue its manœuvre. At any time during such a period of rest, a stimulus of any kind will immediately determine a re-commencement of the manœuvre.

It will be perceived that as soon as the position just described has been attained, gravity, which had hitherto been acting in opposition to the righting movement, now begins to favour that movement. It might, therefore, be anticipated that the *Echinus* would now simply let go all its attachments, and allow itself to roll over into its natural position. But an *Echinus* will never let go its attachments without some urgent reason; and in this case it lets itself down almost as slowly as it raised itself up. So gently, indeed, is the downward movement effected, that an observer can scarcely tell the precise moment at which the righting is concluded. Therefore, in the downward movement, the feet, which at the earlier part of the manœuvre were employed successively in rearing the globe upon its equator, are now employed successively in preventing its too rapid descent (Plate 83, fig. 26).

Several interesting questions arise with reference to these righting movements of *Echinus*. First of all we are inclined to ask what it is that determines the choice of the rows of feet which are delegated to effect the movements. As the animal has a geometrical form of perfect symmetry, we might suppose that when it is placed upon its pole, all the five rows of feet would act in antagonism to one another; for there seems nothing more to determine either the action or the inaction of one row rather than another. The answer to this question is not very clear. First of all it occurred to us that, although the form of the animal presents a geometrical symmetry, the anatomy of the animal is not symmetrical, and therefore that some of the feet-rows might be functionally prepotent over the others. But on observing a great number of specimens, we satisfied ourselves that among different individuals any homologous rows of feet might be used indiscriminately—*i.e.*, taking the madreporic plate as the point of reference, we found that in different individuals rotation might take place in any direction with reference to that plate indifferently. On the other hand, individual specimens would sometimes manifest a marked tendency to rotate in one direction,

i.e., they would repeatedly choose the same feet-rows wherewith to execute their righting movements. In these individual specimens, therefore, the probability is that the feet-rows thus selected were selected because of some slight accidental prepotency or superiority over the others; and thus the explanation in all cases doubtless is that, although the physiological conditions are pretty nicely balanced, they are not so nicely balanced as to leave positively nothing to determine which rows of feet shall be used.

Another question of still more interest is that as to the prompting cause of all these laborious movements. Is it that the animal has some dim consciousness of discomfort, owing to a disturbance of a nascent sense of gravity? Or is the whole series of movements purely mechanical, and determined only by the fact that the feet in the feet-rows are all arranged serially, and therefore when feet A, B, and C have established a firm hold and thereby tilted the globe over a certain distance, opportunity is afforded for D, E, and F to establish a hold, and so on? This question had better, for the present, be deferred.

(B.) *Spines.*—(C.) *Lantern.*—It is, of course, well known that the spines of the *Echinus* are used in locomotion; but hitherto their action does not seem to have been carefully observed, and we are not aware that the part played by the lantern has ever been observed at all. Observations on these points may best be made by taking the animal out of the water, and placing it upon a table; it will then soon begin to walk in some definite direction—*i.e.*, in a straight line—and in doing so the only organs used for the purposes of locomotion are the spines and the lantern, the ambulacral feet under these circumstances not being protruded at all. The rate of locomotion is very slow, viz., about 1 inch per minute; but it is continuous, takes place, as already observed, in a definite direction, and is accomplished by means of a number of highly co-ordinated movements. The latter are as follows:—

The whole dental apparatus, or lantern, admits of being protruded and retracted; when protruded, the sharp and polished point which is composed by the mutual contact of the five teeth, stands out below the ventral surface of the animal; when retracted, this point is drawn within the body cavity of the animal. The movements of protrusion and retraction are perfectly rythmical, at the rate of three or four revolutions per minute. When the lantern is drawn back to its fullest extent, it is tilted to one side, in such a way that the teeth point towards the direction of advance. The lantern is then brought down and protruded till the teeth rest upon the table; some of the spines have meanwhile been rotating on their ball and socket-joints, in such a way that their points are in a position on the table to push the animal towards the direction in which the teeth are pointing. This push being communicated by the spines while the teeth are held firmly down, the result is to raise the whole animal upon the point of its teeth, and to let it fall again upon the other side of the teeth; the point of the teeth is thus used as a fulcrum, over which the animal is made to move by the co-ordinated action of its spines. Of course, when it has completed this movement, the teeth are pointing away from the direction of advance—the whole

lantern having been, as it were, left behind by the movement of the shell over it as a pivot, and therefore sloping away from the direction of advance at the same angle as that with which it had previously sloped towards it. The lantern is now again retracted, and during its retraction, partly rotated upon its horizontal axis, so that by the time it is again protruded, its vertical axis is again pointing towards the direction of advance. And so the manœuvre is repeated over and over again—the *Echinus* advancing by a succession of jerks as it repeatedly tumbles over its teeth. As already stated, the movements of the lantern are rythmical, and therefore the jerks take place at regular intervals. It is important, however, to observe that although the lantern is thus used to assist the spines in locomotion, it is doubtful whether such is the full explanation of the lantern's movements. For, on the one hand, it is certain that these movements are not necessary, but only accessory to locomotion; and, on the other hand, they continue to take place under circumstances where they can be of no use in locomotion. Thus we have observed that young specimens of *Echinus* do not use their lanterns for locomotion, as older specimens always do. Probably the older, and therefore larger specimens, use their lanterns more than the younger and smaller ones, on account of having to move so much heavier a mass. Also, the relatively greater length of the spines in the smaller specimens makes it much more difficult for the lantern to touch the table. This view is confirmed by the fact that, on cutting the spines of young specimens shorter, these small *Echini* begin to use their lanterns after the manner of larger specimens. But, be this as it may, the fact that in young specimens the lanterns rarely touch the table is proof that the spines are here alone sufficient to produce locomotion. Again, as before observed, these peculiar movements of the lantern take place under circumstances where they can be of no use in producing locomotion. Thus, for instance, they take place in young specimens in the same way as in old, although, as just stated, the lanterns in this case do not touch the table at all. And again, if an *Echinus* be placed on its aboral pole, the lantern at once begins its rhythmical movements, and continues them as long as the animal remains in that position. In this case there is added to the movements already described another perfectly rhythmical movement, which consists in closing and opening the teeth—the time of complete closure corresponding with that of greatest protusion, and the time of fullest opening with that of greatest retraction. It appears, therefore, that these rhythmical movements of the lantern, although undoubtedly of use in assisting locomotion in some cases, may possibly have some other function to perform in the economy of the animal. Whether this is so or not, there seems to be some intimate connexion between these movements of the lantern and the movements of the spines; for when one stops the other stops, and when one begins the other begins. The movements of the lantern may best be studied by taking away the top of the *Echinus* shell, and looking down upon the lantern from above; it may then be seen exhibiting its rhythmical movements, which when thus viewed forcibly remind one of the rolling of a ship at sea. The complex muscular system of the

lantern seems to us mainly subservient to the execution of these movements, and yet —so far as we could detect—they are utterly useless for any purpose other than that of assisting locomotion.

(D.) *Pedicellariæ.*—A good deal of speculation has been expended on the probable function of these organs. Prolonged observation has satisfied us that they have a function which has not hitherto been suspected, viz., that of assisting locomotion. A full account of our observations on these structures, however, had better be reserved for the next section of this paper.

3. SPATANGUS.—This animal crawls about somewhat slower than *Echinus*, keeping its very long spines partly erect to act as feelers. It does not appear able to climb perpendicular surfaces. When placed upon its back, it has more difficulty in righting itself than any of the Echinoderms that we have observed; for, on account of its having such flat poles and such short ambulacral feet, it is, when inverted, placed at even a greater disadvantage than is *Echinus*. Therefore, many specimens—especially large specimens—are never able, when inverted on a hard flat surface, to right themselves at all; smaller specimens, however, are able to do so after an expenditure of much time and energy. Their method of doing so is quite different from that of *Echinus*. Indeed, looking to the shape of *Spatangus* and to the character of its pedicels, the method of righting adopted by *Echinus* would be here clearly impossible. This animal, therefore, rights itself entirely by the action of the only organs which are available for the purpose, viz., its long and mobile spines. The long spines are not very many in number; but as their strength and co-ordination is surprising, they enable the animal, by a series of pushings and proppings, eventually to turn itself completely over from one of its flattened surfaces to the other. In doing this it usually, but not invariably, turns over upon its broad end. When the long spines are removed, the animal, of course, is no longer able to right itself.

4. HOLOTHURIANS.—Very little has to be said on these sluggish members of the Echinoderm group. They crawl slowly, and indulge in prolonged periods of quiescence. They are, however, able to climb perpendicular surfaces.

From this account of the natural movements exhibited by the several groups of Echinodermata here considered, it may be observed that we have presented to our view an interesting series of graduated modifications. At one end of this series we have *Echinus* and *Spatangus* with their rays all united into a box-like rigid shell. At the other end of the series we have the Brittle-stars with their muscular rays, highly mobile, and indeed snake-like in their well co-ordinated movements. Midway in the series we have the Sea-cucumber and common Star-fish, where the body is flexible and mobile, though not so much so as in the Brittle-stars. Now, the interesting point to observe is, that in correlation with this graduated difference in the function of the rays, we have a correspondingly graduated difference in the development of the ambulacral system. In *Echinus* and *Spatangus* this system is seen in its most

elaborate and efficient form—in *Echinus* the pedicels, spines, and pedicellariæ being more highly developed and useful than in any of the other groups, except *Spatangus*, where the spines are even more so. In the common Star-fish, Sun-stars, and Sea-cucumbers, the ambulacral feet are still the most important organs of locomotion, although even here we begin to see that the development of the general muscular system has begun to tell upon that of these specially locomotor organs. Again, in *Astropecten* the still greater development of the general muscular system has told still further upon that of the ambulacral feet, the terminal suckers having become aborted. Lastly, the Brittle-stars have altogether discarded the use of their ambulacral feet in favour of the much more efficient organs of locomotion supplied by their muscular rays; and not only the terminal suckers of these feet, but even the whole of the feet themselves, have dwindled into useless rudiments.

§ II. Stimulation.

1. General facts of stimulation.—All the Echinoderms we have observed respond to all kinds of stimulation. The period of latency varies considerably in different species, and in different parts of the same animal. In the Holothurians it is remarkably long, and from the seat of stimulation there very frequently starts a wave of strong contraction, which passes with extreme slowness throughout the length of the animal in the form of a deep constriction. Similar waves frequently occur spontaneously.

All the Echinoderms seek to escape from injury. Thus, for instance, if a Star-fish or *Echinus* is advancing continuously in one direction, and if it be pricked or cut in any part of an excitable surface facing the direction of advance, the animal immediately reverses that direction; or, if it be taken out of the water and a drop of some irritating fluid be placed on any part of the external surface, the animal will endeavour to move away from the source of irritation; whether placed upon a dry table or returned to the water, the Echinoderm will at once strike off in a perfectly straight line from the source of irritation, and for a long time will travel much more rapidly than usual. When two points of the surface are thus irritated, the direction of advance is usually the diagonal between them. When a greater number of points are irritated, the direction of advance becomes uncertain, but if any, even short, interval of time is allowed to elapse between the application of successive stimuli to different parts of the surface, the direction of advance will be in a straight line from the stimulus applied latest. When a Star-fish is fastened upon a perpendicular surface, and any part of its body is irritated, as, *e.g.*, by a nip with the forceps, the animal, if a Sun-star, will actively run away from the irritation. If, however, the latter be followed up and repeated, the Star-fish seems to make up its mind to escape in a still more expeditious manner, for it immediately lets go its hold with all its suckers, and falls to the bottom of the water. A common Star-fish will generally resort to this

method when first irritated, without waiting for a repetition of the stimulus. An *Echinus*, on the other hand, will not drop off a perpendicular surface unless compelled to do so by serious irritation; it crawls away as quickly as possible, and sometimes rotates upon its axis in a manner afterwards to be described, whereby, without leaving go its hold of the perpendicular surface, it is able to alter its position rapidly. But of all the Echinoderms the most curious to observe in this connexion are the Brittle-stars, for these may be made to leap about in any number of directions with much activity, by gently stimulating different parts of their bodies successively. When any part of the dorsal surface of any Star-fish is irritated, not unfrequently one of the arms is doubled over and touches the seat of irritation, as if to endeavour to brush away the offending body.

That the external surface of a Star-fish should prove itself to be excitable is what we should perhaps expect *à priori*, although we might not expect to find so high a degree of co-ordination manifested by the nervous system as is implied by its responses to the cutaneous excitations above mentioned. But that the external surface of an *Echinus* or *Spatangus* should be so highly excitable as it is, we should scarcely have anticipated—particularly before our observation of the external nervous plexus; for at first sight it would seem that the numberless long and mobile feet—to say nothing of the spines—would be sufficient to convey all the information that the animal requires concerning the external world, without its exterior requiring to be rendered sensitive over its whole surface. Yet we find, so far is this from being the case, that the external surface cannot be touched with a needle's point at any part without the whole animal being affected thereby. We have already described the nervous plexus whereby this general sensitiveness of the external surface is secured. We must now enter pretty fully into the functions of this plexus as revealed by sundry experiments on the multitudinous and wonderful system of organs which, either directly or indirectly, depend upon this plexus for their innervation.

These organs are the ambulacral feet, the spines, and the pedicellariæ. That all these organs are in nervous connexion with the external plexus is proved by the fact that when any part of the external surface is touched, however gently, all the feet, spines, and pedicellariæ within reach of that point, and even far beyond, immediately approximate and close in upon the point, so holding fast to the needle, or whatever other body may be used as the instrument of stimulation. This simultaneous movement of such a little forest of prehensile organs is a singularly beautiful spectacle to witness. In executing it, the pedicellariæ are much the most active, the spines somewhat slower, and the ambulacral feet very much slower. If the object with which the external surface is touched be itself small enough, or presents edges narrow enough, to admit of the forceps on the pedicellariæ establishing a hold upon it, it is seen to be immediately seized by some of these organs, and held there till the spines and ambulacral feet come up to assist; but if the object is too large, or does not present any surfaces which the pedicellariæ are able to catch—such, for instance, as

the point of a pencil—the spines alone are able to hold it with wonderful firmness by forcing their tips against it on all sides.

The area thus affected by an ordinary stimulation, such as that supplied by a touch with a needle, measures in a longitudinal direction about half an inch. The extent of the area affected in a tranverse or latitudinal direction depends upon the point stimulated with reference to the ambulacral feet. Midway in an interambulacral area the influence extends as far as the double rows of feet on either side; the feet, however, of the inner, or nearer rows, moving more decidedly than those of the outer, or further, rows. The spines are rarely affected beyond the area named by a stimulus of mere contact, but in the case of the pedicellariæ the irradiation of the stimulating influence may proceed further, sometimes extending as far as the second double row of feet, or ambulacral area, on either side; the certainty and activity of their movements, however, rapidly diminish with their distance from the seat of stimulation. At and near the seat of stimulation, *i.e.*, within the area first named, the certainty and activity of their movements are very great, and the period of latency very short; in other words, immediately any solid body touches any part of the external surface of an *Echinus*, it is surrounded by all the pedicellariæ in the neighbourhood, while even those which are too far away to touch the object will, perhaps for the long distance round which we have named, bend towards it.

2. Physiology of the Pedicellariæ.—And here we have the proof of the function of the pedicellariæ. In climbing perpendicular or inclined surfaces of rock, covered with waving sea-weeds, it must be of no small advantage to an *Echinus* to be provided on all sides with a multitude of forceps, all mounted on movable stalks, which instantaneously bring their grasping forceps to bear upon and to seize a passing frond. The frond being thus arrested, the spines come to the assistance of the pedicellariæ, and both together hold the *Echinus* to the support furnished by the sea-weed. Moreover the sea-weed is thus held steady till the ambulacral feet have time also to establish their hold upon it with their sucking discs. That the grasping and arresting of fronds of sea-weed in this way for the purposes of locomotion constitute an important function of the pedicellariæ, may at once be rendered evident experimentally by drawing a piece of sea-weed over the surface of a healthy *Echinus* in the water. The moment the sea-weed touches the surface of the animal, it is seen and felt to be seized by a number of these little grasping organs, and—unless torn away by a greater force than is likely to occur in currents below the surface of the sea—it is held steady till the ambulacral suckers have time to establish their attachments upon it. Thus there is no doubt that the pedicellariæ are able efficiently to perform the function which we regard as their chief function. We so regard this function, not merely because it is the one that we observe these organs chiefly to perform, but also because we find that their whole physiology is adapted to its performance. Thus their multitudinous number and ubiquitous situation all over the external surface of the animal, is suggestive of their being adapted to catch something which may come upon them from any side,

and which may have strings and edges so fine as to admit of being enclosed by the forceps. Again, the instantaneous activity with which they all close round and seize a moving body of a size that admits of their seizing it, is suggestive of the objects which they are adapted to seize being objects which rapidly brush over the surface of the shell, and therefore objects which, if they are to be seized at all, must be seized instantaneously. Lastly, we find, on experimenting upon pedicellariæ whether *in situ* or when separated from the *Echinus*, that the clasping action of the forceps is precisely adapted to the function which we are considering; for not only is the force exerted by the forceps during their contraction of an astonishing amount for the size of the organ (the serrated mandibles of the trident pedicellariæ holding on with a tenacity that can only have reference to some objects liable to be dragged away from their grasp), but it is very suggestive that this wonderfully tenacious hold is spontaneously relaxed after a minute or two. That is to say, the pedicellariæ tightly fix the object which they have caught for a time sufficient to enable the ambulacral suckers to establish their connexions with it, and then they spontaneously leave go; their grasp is not only so exceedingly powerful while it lasts, but it is as a rule timed to suit the requirements of the pedicels.*

On the whole, therefore, we can entertain but little doubt concerning the main function at least of the trident pedicellariæ in the *Echinus*. But criticism will, of course, immediately object that in other Echinoderms these organs are too small or too few to be of any use in assisting locomotion in the way just described. The only answer to this objection is, that in ascertaining the function of any organ it is safest to study the activities of that organ in its most developed, or least degenerated, form. We could not, for instance, ever ascertain the function of the spines in any of the Echinodermata, if we were to consider these structures only in the Star-fishes and Holothurians, and if the pedicellariæ seem to be so small in Star-fish as not to appear capable of performing the function here assigned to them in *Echinus*, the explanation probably is that, as in the analogous cases of the spines, changed habits of life on the part of the animals have caused these inherited appendages to dwindle from disuse. Thus, for instance, Brittle-stars never climb sea-weed-covered-rocks at all, and those Star-fish which do so have their ambulacral feet restricted to the ventral surface; it would therefore be useless for these animals to have well-developed pedicellariæ, adapted to hold sea-weeds steady in the manner which may be of so much use to the globular *Echinus*, who throws out on all sides feet feeling for attachments. Therefore, whether the pedicellariæ of these other Echinoderms perform any function that yet remains to be detected, or whether they are mere rudiments now useless, we think

* When pedicellariæ are detached from the *Echinus*, however, it is frequently observable that their grasp becomes, as it were, spasmodic, and endures for an indefinite time. For instance, it is not unusual to see a pedicellaria, which has been torn from its root while clutching a pedicel, carried about holding on to the pedicel for a very long time. But this spasmodic or continuous grasp of the organ when severed we have not observed to occur when the organ is *in situ*.

that the presence of such organs in these other Echinodermata raises no real difficulty in the way of accepting the proof which we have rendered of their observed functions as they occur in their most efficient forms.

Concerning the physiology of the pedicellariæ little further remains to be said under the present section. It may be stated, however, that the mandibles, which are constantly swaying about upon their contractile stalks as if in search for something to catch, will snap at an object only if it touches the inner surface of one or more of the expanded mandibles. Moreover, in the larger pedicellariæ, a certain part of the inner surface of the mandibles is much more sensitive to contact than is the rest of that surface; this part is a little pad about one-third of the way down the mandible (Plate 80, fig. 13, *s*); a delicate touch with a hair upon this part of any of the three mandibles is certain to determine an immediate closure of all the three. It is obvious that there is an advantage in the sensitive area, or zone, being placed thus low down in the length of the mandibles to ensure that the whole apparatus will not close upon an object till the latter is far enough within the grasp of the mechanism to give this mechanism the best possible hold. If, for instance, the tips of the mandibles were the most sensitive parts, or even if their whole inner surfaces were uniformly sensitive, the apparatus would be constantly closing upon objects when these merely brushed past their tips, and therefore closing prematurely for the purpose of grasping. But, as it is, the apparatus is admirably adapted to waiting for the best possible chance of getting a secure hold, and then snapping upon the object with all the quickness and tenacity of a spring-trap.

Another point worth mentioning is that if, after closure, any one or more of the mandibles be gently stroked on its outer surface near the base, all the mandibles are by this stimulation usually, though not invariably, induced again to expand. This is the only part of the whole organ the stimulation of which thus exerts an inhibitory influence on the contractile mechanism. If there is any functional purpose served by this relaxing influence of stimulating this particular part of the apparatus, we think it can only be as follows. When a portion of sea-weed brushes this particular part, it must be well below the tips of the mandibles, and therefore in a position where it, or some over-lying portion, may soon pass between the mandibles, if the latter are open; hence when touched in this place the mandibles, if closed, open to receive the sea-weed, should any part of it come within their cavity.

3. Physiology of the Spines.—We must next consider stimulation with reference to the spines. It has already been said that these organs co-operate with the pedicellariæ in grasping any instrument of stimulation, and this proves that for a certain area round any seat of stimulation the spines admit of co-ordinated action. Further experiments prove that there is no limit to the area within which co-ordinated action of the spines may take place; but that all the spines of the organism may work together to the attainment of some common end. Thus it has already been stated in a previous part of this paper that a *Spatangus*, when placed upon its back, is able to right itself by

the co-ordinated action of its spines alone; and also that an *Echinus*, when taken out of the water and placed upon a table, will walk in a determinate direction by the same means. The very complete co-ordination of the spines implied by these facts is, however, rendered still more conspicuous by experiments in stimulation; for if, while an *Echinus* is walking on the table in the manner just alluded to, a scrape with a scalpel, a drop of spirit, a lighted match, or any other severe stimulus be applied at some one part of the animal's exterior, the spines all over the surface begin to take on an active bristling movement, and the direction of advance is immediately changed into a straight line of escape from the source of injury. And, were it necessary, other experiments could be detailed to show that the multitudinous spines of an *Echinus* are as closely co-ordinated in their action as so many limbs. To this account of the physiology of the spines it may be added that the nervous plexus [illegible] the tubercles on which they [illegible] more sensitive to stimulation than any other part of the external plexus. This is shown by the fact that, if the tubercle is stimulated by enclosing the spine in the tube of a pipette, and pressing the tubercle with the glass edges of the latter, more activity and a greater extent of irradiation of the stimulus among the spines and pedicellariæ is observed, than when any other part of the surface is similarly stimulated.

4. Detailed facts of stimulation.—At the commencement of this section it has already been stated, as a general fact, that when two points of the surface of an Echinoderm are irritated, the direction of advance which results from their joint influence is usually the diagonal between the two; also that, "when a greater number of points are irritated, the direction of advance becomes uncertain;" and lastly, that "if any, even short interval of time is allowed to elapse between the application of successive stimuli to different parts of the surface, the direction of advance will be in a straight line away from the stimulus applied latest." The following more detailed observations on this subject may here be worth recording.

Echini actively crawling in water along the floor of a tank were the subjects of the experiments, which are thus recorded in our notes:—

"1. Cut off tips of spines facing direction of advance—no effect.

"2. Cut off tips of protruded feet facing direction of advance—all the rest of the row retracted, animal stopped for some minutes, and then proceeded in the same direction as before.

"3. Plucked out some pedicellariæ facing direction of advance—no effect.

"4. Scraped with a needle small portion of the surface facing direction of advance—animal immediately stopped and reversed its direction.

Injuries 1, 2, 3, and 4, were inflicted on the equator.

"5. Scraped equator with a scalpel on two points opposite to each other—animal crawled at right angles to the line of injury.

"6. Scraped similarly at the aboral pole—no effect; there was no reason why injury here should determine escape in one direction rather than in another.

"7. Scraped similarly near pole, and half-way between pole and equator—little or no effect.

"8. Scraped in rapid succession five equatorial injuries, one on each of the five inter-ambulacral spaces—*Echinus* crawled actively in one determinate direction; the equal and equidistant injuries all round the animal neutralized each other.

"9. Scraped a band of uniform width all the way round the equator—same result as in 8.

"10. Band of injury in 9 widened on the side facing direction of advance—no effect. Still further widened—slight change of direction, and, after a little time, persistent crawling away from widest part of injured zone.

Repeated experiment on other specimens, scraping round whole equator and simultaneously making one part of the zone of injury wider than the rest—same result; the animal crawled away from the *greatest amount* of injury.

"11. Scraped on base side of equator facing direction of advance—immediate reversal of that direction.

"12. After a few minutes similarly scraped opposite side—direction of advance immediately reversed to original one.

"13. Similarly scraped midway between the two previous injuries—direction of advance became oblique between the two first injuries, with a considerable simultaneous rotation upon the vertical axis of the animal.

"14. Similarly scraped a number of places on all aspects of the animal indiscriminately—direction of advance became uncertain and discontinuous, with a strong tendency to rotation upon vertical axis."

5. Physiology of the Pedicels.—Taking here the Star-fish as a type of the Echinodermata, the results of our experiments on this head, and so far as stimulation is concerned, are as follows. When a drop of acid, or other severe stimulation is applied to any part of a row of protruded pedicels, that whole row is immediately retracted, the pedicels retracting successively from the seat of irritation—so that if the latter be in the middle point of the series, two series of retractions are started, proceeding in opposite directions simultaneously; the rate at which they travel is rather slow. This process of retraction, however, although so complete within the ray irritated, does not extend to the other rays. But if the stimulus be applied to the centre of the disc, upon the oral surface of the animal, all the feet in all the rays are more or less retracted—the process of retraction radiating serially from the centre of stimulation. The influence of the stimulus, however, diminishes perceptibly with the distance from the centre; thus, if weak acid be used as the irritant, it is only the feet near the bases of the rays that are retracted; and even if very strong acid be so used, it is only the feet as far as one-half or two-thirds of the way up the rays that are fully retracted—the remainder only having their activity impaired, while those near the tip may not be affected at all. If the drop of acid be placed on the dorsal, instead of the ventral

surface of the disc, the effect on the feet is found to be just the converse; that is, the stimulus here applied greatly increases the activity of the feet. Further experiments show that this effect is produced by a stimulus applied anywhere over the dorsal aspect of the animal; so that, for instance, if a drop of acid be placed on the skin, at the edge of a ray, and therefore just external to the row of ambulacral feet, the latter will be stimulated into increased activity; whereas, if the drop of acid had been placed a very small distance past the edge of the ray, so as to touch some of the feet themselves, then the whole row would have been drawn in. We have here rather an interesting case of antagonism, which is particularly well marked in *Astropecten*, on account of the active writhing movements which the feet exhibit when stimulated by an irritant placed on the dorsal surface of the animal. It may be added that in this antagonism the inhibitory function is the stronger; for when the feet are in active motion, owing to an irritant acting on the dorsal surface, they may be reduced to immediate quiescence—*i.e.*, retracted—by placing another irritant on the ventral surface of the disc. Similarly, if retraction has been produced by placing the irritant on the ventral surface of the disc, activity cannot be again induced by placing another drop of the irritant on the dorsal surface.

6. Luminous stimulation.—The only other observations we have to detail under the present section are those relating to the influence of light. We have found unequivocal evidence of the Star-fish (with the exception of the Brittle-stars) and the *Echini* manifesting a strong disposition to crawl towards, and to remain in, the light. Thus if a large tank be completely darkened, except at one end where a narrow slit of light is admitted, and if a number of Star-fish and *Echini* be scattered over the floor of the tank, in a few hours the whole number, with the exception of perhaps a few per cent., will be found congregated in the narrow slit of light. The source we used was diffused daylight, which was admitted through two sheets of glass, so that the thermal rays might be considered practically excluded. The *intensity* of the light which the Echinoderms are able to perceive may be very feeble indeed; for in our first experiments we boarded up the face of the tank with ordinary pine-wood, in order to exclude the light over all parts of the tank except at one narrow slit between two of the boards. On taking down the boards we found indeed the majority of the specimens in or near the slit of light; but we also found a number of other specimens gathering all the way along the glass face of the tank that was immediately behind the pine-boards. On repeating the experiment with blackened boards, this was never found to be the case; so there can be no doubt that in the first experiments the animals were attracted by the faint glimmer of the white boards, as illuminated by the very small amount of light scattered from the narrow slit through a tank all the other sides of which were black slate. Indeed, towards the end of the tank, where some of the specimens were found, so feeble must have been the intensity of this glimmer, that we doubt whether even human eyes could have descried it very distinctly. Owing to the prisms at our command not having sufficient dispersive power for the experiments, and

not wishing to rely on the uncertain method of employing coloured glass, we were unable to ascertain how the Echinoderms might be affected by different rays.

On removing with a pointed scalpel the eye-spots from a number of Star-fishes and Echini without otherwise injuring the animals, the latter no longer crawled towards the light, even though this were admitted to the tank in abundance; but they crawled promiscuously in all directions. On the other hand, if only one of the five eye-spots were left intact, the animals crawled toward the light as before.

§ III. Section.

1. Star-fish.—Single rays detached from the organism crawl as fast and in as direct a direction as do the entire animals. They also crawl towards the light, up perpendicular surfaces, and sometimes away from injuries; but they do not invariably, or even generally, seek to escape from the latter, as is so certain to be the case with entire animals. Lastly, when inverted, separated rays right themselves as quickly as do the unmutilated organisms.

Removing the tip of a severed ray does not impair any of these movements, except, of course, the crawling towards light, which it completely destroys. Dividing the nerve in any part of its length has the effect, whether or not the ray is detached from the animal, of completely destroying all physiological continuity between the pedicels on either side of the line of division. Thus, for instance, if the nerve be cut across half-way up its length, the row of pedicels is at once physiologically bisected, one-half of the row becoming as independent of the other half as it would were the whole ray divided into two parts; that is to say, the distal half of the row may crawl while the proximal half is retracted, or *vice-versâ*, and if a drop of acid be placed on either half, the serial contraction of the pedicels in that half stops abruptly at the line of nerve-division. As a result of this complete physiological severance, when a detached ray so mutilated is inverted, it experiences much greater difficulty in righting itself than it does before the nerve is divided. The line of nerve-injury lies flat upon the floor of the tank, while the central and distal portions of the ray—*i.e.*, the portions on either side of that line—assume various movements and shapes. The central portion is particularly apt to take on the form of an arch, in which the central end of the severed ray and the line of nerve-section constitute the points of support (tetanus ?) (Plate 83, fig. 27), or the central end may from the first show paralysis, from which it never recovers. The distal end, on the other hand, usually continues active, twisting about in various directions, and eventually fastening its tip upon the floor of the tank to begin the spiral movement of righting itself (Plate 83, fig. 27). This movement then continues as far as the line of nerve-injury, where it invariably stops (Plate 83, fig. 27). The central portion may then be dragged over into the normal position, or may remain permanently inverted, according to the strength of pull exerted by the distal portion; as a rule, it does not itself assist in the righting movement, although its feet usually continue protruded and mobile.

The above observations have reference to the common Star-fish, but they apply equally to other Star-fishes, except that in *Astropecten* single detached rays are not able to right themselves when inverted (owing to the feet not being used by this species for this purpose, and to the other rays being absent), and that after division of the nerve in a ray of this species, the feet of the proximal portion usually manifest more activity than those of the distal. The destruction, however, of physiological continuity between the two portions is as complete as in the case of the common Star-fish. Single detached rays of Brittle-stars are able when inverted to right themselves; they wriggle round by means of their snake-like movements, and do not require, as is the case with the less active rays of *Astropecten*, the assistance of adjacent rays to effect the manœuvre. On the whole, then, it may be said, as a general statement, that in all the species of Star-fish which we have observed, the effect of a transverse section of the nerve in a ray is that of completely destroying physiological continuity between the pedicels on either side of the section.

The only other experiments in nerve-section to which the simple anatomy of a Star-fish exposes itself is that of dividing the nerve-ring in the disc; or, which is virtually the same thing, while leaving this intact, dividing all the nerves where they pass from it into the rays. In specimens mutilated by severing the nerves at the base of each of the five rays, or by dividing the nerve-ring between each ray, the animal loses all power of co-ordination among its rays. When a common Star-fish is so mutilated it does not crawl in the same determinate manner as an unmutilated animal, but, if it moves at all, it moves slowly and in various directions. When inverted, the power of effecting the righting manœuvre is seen to be gravely impaired, although eventually it is always accomplished. There is a marked tendency, as compared with unmutilated specimens, to a promiscuous distribution of spirals and doublings, so that instead of a definite plan of the manœuvre being formed from the first, as is usually the case with unmutilated specimens, such a plan is never formed at all; among the five rays there is a continual change of unco-ordinated movements, so that the righting seems to be eventually effected by a mere accidental prepotency of some of the righting movements over others. Appended is a sketch of such unco-ordinated movement, taken from a specimen which for more than an hour had been twisting its rays in various directions (Plate 84, fig. 28). Another sketch is appended to show a form of bending which specimens mutilated as described are very apt to manifest, especially just after the operation. When placed upon their dorsal surface they turn up all their rays with a peculiar and exactly similar curve in each, which gives to the animal a somewhat tulip-like form (Plate 84, fig. 29). This form is never assumed by unmutilated specimens, and in mutilated ones, although it may last for a long time, it is never permanent. In detached rays this peculiar curve is also frequently exhibited; but if the nerve of such a ray is divided at any point in its length, the curve is restricted to the distal portion of the ray; it stops abruptly at the line of nerve-section. When entire Star-fish are mutilated by a section of each nerve-

trunk half-way up each ray, and the animal is then placed upon its back, the tetanic contraction of the muscles in the rays before mentioned as occurring under this form of section in detached rays, has the effect, when now occurring in all the rays, of elevating the disc from the floor of the tank. This opisthotonous-like spasm is not, however, permanent; and the distal ends of the rays forming adhesions to the floor of the tank, the animal eventually rights itself, though much more slowly than unmutilated specimens. After it has righted itself, although it twists about the distal portions of the rays, it does not begin to crawl for a long time, and when it does so, it crawls in a slow and indeterminate manner. Star-fish so mutilated, however, can ascend perpendicular surfaces.

The loss of co-ordination between the rays caused by division of the nerve-ring in the disc is rendered most conspicuous in Brittle-stars, from the circumstance that in locomotion and in righting so much here depends upon co-ordinated muscular contraction of the rays. Thus, for instance, when a Brittle-star has its nerve-ring severed between each ray, an interesting series of events follows. First, there is a long period of profound shock—spontaneity, and even irritability, being almost suspended, and the rays appearing to be rigid, as if in tetanic spasm. After a time, feeble spontaneity returns—the animal, however, not moving in any determinate direction. Irritability also returns, but only for the rays immediately irritated, stimulation of one ray causing active writhing movements in that ray, but not affecting, or only feebly affecting, the other rays. The animal, therefore, is quite unable to escape from the source of irritation, the aimless movements of the rays now forming a very marked contrast to the instantaneous and vigorous leaping movements of escape which are manifested by unmutilated specimens. Moreover, unmutilated specimens will vigorously leap away, not only from stimulation of the rays, but also from that of the disc; but those with their nerve-ring cut make no attempts to escape, even from the most violent stimulation of the disc. In other words, the disc is entirely severed from all physiological connexion with the rays.

If the nerve-ring be divided at two points, one on either side of a ray, that ray becomes physiologically separated from the rest of the organism. If the two nerve-divisions are so placed as to include two adjacent rays—*i.e.*, if one cut is on one side of a ray and the other on the further side of an adjacent ray—then these two rays remain in physiological continuity with one another, although they suffer physiological separation from the other three. When a Brittle-star is completely divided into two portions, one portion having two arms and the other three, both portions begin actively to turn over on their backs, again upon their faces, again upon their backs, and so on alternately for an indefinite number of times. These movements arise from the rays, under the influence of stimulation caused by the section, seeking to perform their natural movements of leaping, which however end, on account of the weight of the other rays being absent, in turning themselves over. An entire Brittle-star when placed on its back after division of its nerve-ring is not able to right itself, owing to

the destruction of co-ordination among its rays. *Astropecten*, under similar circumstances, at first bends its rays about in various ways, with a preponderant disposition to the tulip form, and keeps its ambulacral feet in active movement. But after half an hour, or an hour, the feet generally become retracted and the rays nearly motionless—the animal, like a Brittle-star, remaining permanently on its back. In this, as in other species, the effect of dividing the nerve-ring on either side of a ray is that of destroying its physiological connexion with the rest of the animal, the feet in that ray, although still remaining feebly active, no longer taking part in any co-ordinated movement—that ray, therefore, being merely dragged along by the others.

Under this division it only remains further to be said, that section of the nerve-ring in the disc, or the nerve-trunks of the rays, although as we have seen so completely destroying physiological continuity in the rows of ambulacral feet and muscular system of the animal, does not destroy physiological continuity in the external nerve plexus; for however much the nerve-ring and nerve-trunks may be injured, stimulation of the dorsal surface of the animal throws all the ambulacral feet and all the muscular system of the rays into active movement. This fact proves that the ambulacral feet and the muscles are all held in nervous connexion with one another by the external plexus, without reference to the integrity of the main nerve-trunks.

2. Echini.—(A.) *Section of external surface of shell.*—If a cork-borer be applied to the external surface of the shell of an *Echinus*, and rotated there till the calcareous substance of the shell is reached, and therefore a continuous circular section of the overlying tissues effected, it is invariably found that the spines and pedicellariæ within the circular area are physiologically separated from the contiguous spines and pedicellariæ, as regards local reflex excitability. That is to say, if any part of this circular area be stimulated, all the spines and pedicellariæ within that area immediately respond to the stimulation in the ordinary way; while none of the spines or pedicellariæ surrounding the area are affected. Similarly, if any part of the shell external to the circumscribed area be stimulated, the spines and pedicellariæ within the area are not affected. These facts prove that the function which is manifested by these appendages, of localising and gathering round a seat of stimulation, is exclusively dependent upon the external nerve plexus. It is needless to add that in this experiment it does not signify of what size or shape or by what means the physiological island is made, so long as the destruction of the nervous plexus by a closed curve of injury is rendered complete. In order to ascertain whether, in the case of an unclosed curve of injury, any irradiation of a stimulus would take place round the ends of the curve, we made sundry kinds of section. It is, however, needless to describe these, for they all showed that, after injury of a part of the plexus, there is no irradiation of the stimulus round the ends of the injury. Thus, for instance, if a short straight line of injury be made, by drawing the point of a scalpel over the shell, say along the equator of the animal, and if a stimulus be afterwards applied on either side of that line, even quite close to one of its ends, no effect will be exerted on the spines or

pedicellariæ on the other side of the line. This complete inability of a stimulus to escape round the ends of an injury, forms a marked contrast to the almost unlimited degree in which such escape takes place in the more primitive nervous plexus of the Medusæ.

Although the nervous connexions on which the spines and pedicellariæ depend for their function of localising and closing round a seat of stimulation are thus shown to be completely destroyed by injury of the external plexus, other nervous connexions, upon which another function of the spines depends, are not in the smallest degree impaired by such injury. The other function to which we allude is that which brings about the general co-ordinated action of all the spines for the purposes of locomotion. That this function is not impaired by injury of the external plexus is proved by the fact that, if the area within a closed line of injury on the surface of the shell be strongly irritated, all the spines over the whole surface begin to manifest their peculiar bristling movements, and by this co-ordinated action rapidly move the animal in a straight line of escape from the source of irritation; the injury to the external plexus, although completely separating the spines enclosed by it from their neighbouring spines as regards what may be called their local function of seizing the instrument of stimulation, nevertheless leaves them in undisturbed connexion with all the other spines in the organism as regards what may be called their universal function of locomotion.

(B.) Evidently, therefore, this more universal function must depend upon some other set of nervous connexions; and experiment shows that these are distributed over all the *internal* surface of the shell. Our mode of experimenting was to divide the animal into two hemispheres, remove all the internal organs of both hemispheres (these operations producing no impairment of any of the functions of the pedicels, spines, or pedicellariæ), and then paint with strong acid the inside of the shell—completely washing out the acid after about a quarter of a minute's exposure. The results of a number of experiments conducted on this method may be thus epitomised:—

The effect of painting the back or inside of the shell with strong acid (*e.g.*, pure HCl) is that of at first strongly stimulating the spines into bristling movements, and soon afterwards reducing them to a state of quiescence, in which they lie more or less flat, and in a peculiarly confused manner, that closely resembles the appearance of corn when "laid" by the wind. The spines have now entirely lost both their spontaneity and their power of responding to a stimulus applied on the external surface of the shell—*i.e.*, their local reflex excitability, or power of closing in upon a source of irritation. These effects may be produced over the whole external surface of the shell, by painting the whole of the internal surface; but if any part of the internal surface be left unpainted, the corresponding part of the external surface remains uninjured. Conversely, if all the internal surface be left unpainted except in certain lines or patches, it will only be corresponding lines and patches on the external surface that

suffer injury. It makes no difference whether these lines or patches be painted in the course of the ambulacral feet, or anywhere in the inter-ambulacral spaces.

The above remarks, which have reference to the spines, apply equally to the pedicellariæ, except that their spontaneity and reflex irritability are not destroyed, but only impaired.

Some hours after the operation it usually happens that the spontaneity and reflex irritability of the spines return, though in a feeble degree, and also those of the pedicellariæ, in a more marked degree. This applies especially to the reflex irritability of the pedicellariæ; for while their spontaneity does not return in full degree, their reflex irritability does—or almost in full degree.

These experiments, therefore, seem to point to the conclusions—1st, that the general co-ordination of the spines is dependent on the integrity of an internal nerve-plexus, which, however, we have not been able to detect histologically; 2nd, that the hypothetical internal plexus is everywhere in intimate connexion with the external;* and 3rd, that complete destruction of the former, while profoundly influencing the functions of the latter, nevertheless does not wholly destroy them.

In order that a more clear conception may be rendered of the experiments on which these conclusions are based, we shall here quote from our notes one complete observation :—

"*Echinus* was divided into two hemispheres.

"After evisceration one hemisphere was painted over whole of internal surface with HNO_3. (A.)

"The other was painted down one row of ambulacral plates, and also down the inter-ambulacral plates at another part of internal shell. (B.)

"In (A.) the spines were 'laid'; spontaneity and reflex irritability almost totally destroyed.

"In (B.) similar effects observed above painted areas—unpainted areas unimpaired.

"Three hours after, no considerable recovery where painted; unpainted areas as active as before."

One further point, brought out by further experiments, may here be most conveniently mentioned; it is that a specially great influence, or shock, seems to be exerted on the external plexus by injury of the hypothetical internal plexus *along the lines of the ambulacral pores*. The following observations will serve to show this :—

"Another specimen was divided into two hemispheres. In one hemisphere two adjacent ambulacral rows were thoroughly scraped on internal surface of shell, and

* It is remarkable that painting a portion of the internal surface of the shell should have the effect of injuring the spines and pedicellariæ of the *corresponding* portion of the external surface; for the fact seems to show that there must everywhere be intimate nervous connexions passing through the calcarious substance of the shell. So far, however, we have not been able to detect histological evidence of such connexions.

then well rubbed with sandpaper and brickdust.* The spines along these lines were laid in a very marked way, while spontaneity and reflex irritability, not only along them, *but also in the inter-ambulacral spaces between them*, were completely destroyed. The rest of the hemisphere was normally active.

"Ten minutes after operation the laid spines became more erect, and reflex irritability partly returned.

"Twenty minutes after operation pedicellariæ nearly completely recovered spontaneity and reflex irritability; spines still very imperfectly so.

"Two hours after operation both spines and pedicellariæ of the inter-ambulacral area *completely recovered in all respects*."

(C.) If an *Echinus* is divided into two hemispheres by an incision carried from pole to pole through any meridian, the two hemispheres will live for days, crawling about in the same manner as entire animals; if their ocular plates are not injured, they seek the light, and when inverted they right themselves. The same observations apply to smaller segments, and even to single detached rows of ambulacral feet. The latter are, of course, analogous to the single detached rays of a Star-fish, so far as the system of ambulacral feet is concerned; but looking to the more complicated apparatus of locomotion (spines and pedicellariæ), as well as to the rigid consistence and awkward shape of the segment—standing erect, instead of lying flat—the appearance presented by such a segment in locomotion is much more curious, if not surprising, than that presented by the analogous part of a Star-fish under similar circumstances. It is still more surprising that such a fifth-part segment of an *Echinus* will, when propped up on its ab-oral pole (Plate 84, fig. 30), right itself (Plate 84, fig. 31) after the manner of larger segments or entire animals. They, however, experience more difficulty in doing so, and very often, or indeed generally, fail to complete the manœuvre.

(D.) We are now again face to face with a question already propounded in § II., viz.: Is the action of the ambulacral feet in executing these righting movements of a merely serial kind, or does it depend upon nervous co-ordination? We have found this question very difficult of solution, and in the end have arrived at the conclusion that both principles are combined—the action of the feet being serial, but also assisted by nervous co-ordination. The experiments which lead us to this conclusion are as follows:—

If an unmutilated *Echinus* be suspended by a thread in an inverted position halfway up the side of a tank, in such a way that the ambulacral feet on one side of the ab-oral pole are alone able to reach the perpendicular wall, these feet as quickly as they can establish their attachments to that wall; the thread being then removed,

* This method of destroying the hypothetical plexus was here adopted in preference to the method of painting with acid, in order to avoid a possible source of fallacy in some of the acid passing through the perforations of the shell, and so finding its way over the external surface. All our experiments with acid were on other specimens controlled by similar experiments conducted on this method.

the *Echinus* is left sticking to the side of the tank in an inverted position by means of the ab-oral ends of two adjacent feet-rows (Plate 85, fig. 32). Under these circumstances, as we should expect from the previous experiments, the animal sets about righting itself as quickly as possible. Now, if the righting action of the feet were entirely and only of a serial character, the righting would require to be performed by rearing the animal upwards; the effect of foot after foot in the same rows being applied in succession to the side of the tank, would require to be that of rotating the globular shell against the side of the tank towards the surface of the water, and therefore against the action of gravity. This is sometimes done, which proves that the energy required to perform the feat is not more than a healthy *Echinus* can expend. But much more frequently the *Echinus* adopts another device, and the only one by which it is possible for him to attain his purpose without the labour of rotating upwards: he rotates laterally and downwards in the form of a spiral. Thus, let us call the five feet-rows, 1, 2, 3, 4, and 5 (Plate 85, figs. 33 and 34), and suppose that rows 1 and 2 are in use near their ab-oral ends in holding the animal inverted against the perpendicular side of a tank. The downward spiral rotation would then be effected by gradually releasing the outer feet in row 1, and simultaneously attaching the outer feet in row 2 (*i.e.*, those nearest to row 3, and furthest from row 1), as far as possible to the outer side of that row. The effect of this is to make the globe roll far enough to that side to enable the inner feet of row 3 (*i.e.*, those nearest to row 2), when fully protruded, to touch the side of the tank. They establish their adhesions, and the residue of feet in row 1, now leaving go their hold, these new adhesions serve to roll the globe still further round in the same direction of lateral rotation, and so the process proceeds from row to row; but the globe does not merely roll along in a horizontal direction, or at the same level in the water, for each new row that comes into action takes care, so to speak, that the feet which it employs shall be those which are as far below the level of the feet in the row last employed as their length when fully protruded (*i.e.*, their power of touching the tank) renders possible. The rotation of the globe thus becomes a double one, lateral and downwards, till the animal assumes its normal position with its oral pole against the perpendicular tank wall. So considerable is the rotation in the downward direction, that the normal position is generally attained before one complete lateral, or equatorial, rotation is completed.

The result of this experiment, therefore, implies that the righting movements are due to something more than the merely successive action of the series of feet to which the work of righting the animal may happen to be given. The same conclusion is pointed to by the results of the following experiment.

A number of vigorous *Echini* were thoroughly shaved with a scalpel over the whole half of one hemisphere—*i.e.*, the half from the equator to the oral pole. They were then inverted on their ab-oral poles. The object of the experiment was to see what the Echini which were thus deprived of the lower half of three feet-rows would do when, in executing their righting manœuvres, they attained to the equatorial position

and then found no feet wherewith to continue the manœuvre. The result of this experiment was first of all to show us that the *Echini* invariably chose the unmutilated feet-rows wherewith to right themselves. Probably this is to be explained, either by the general principle to which the escape from injury is due,—viz., that injury inflicted on one side of an Echinoderm stimulates into increased activity the locomotor organs of the opposite side,—or by the consideration that destruction of the lower half of a row very probably induces some degree of shock in the remaining half, and so leaves the corresponding parts of the unmutilated rows prepotent over the mutilated one. Be this as it may, however, we found that the difficulty was easily overcome by tilting the animal over upon its mutilated feet-rows sufficiently far to prevent the unmutilated rows from reaching the floor of the tank. When held steadily in this position for a short time, the mutilated rows established their adhesions, and the *Echinus* was then left to itself. Under these circumstances an *Echinus* will always continue the manœuvre along the mutilated feet-rows with which it was begun, till the globe reaches the position of resting upon its equator, and therefore arrives at the line where the shaved area commences. The animal then remains for hours in this position, with a gradual but continuous motion backwards, which appears to be due to the successive slipping of the spines—these organs in the righting movements being always used as props for the ambulacral feet to pull against while rearing the globe to its equatorial position, and in performing this function on a slate floor the spines are liable often to slip. The only other motion exhibited by *Echini* thus situated is that of a slow rolling movement, now to one side and now to another, according to the prepotency of the pull exerted by this or that row of ambulacral feet. Things continue in this way until the slow backward movement happens to bring the animal against some side of the tank, when the uninjured rows of ambulacral feet immediately adhere to the surface and rotate the animal upwards or horizontally, until it attains the normal position. But if care be taken to prevent contact with any side of the tank, the mutilated *Echinus* will remain propped on its equator for days; it never adopts the simple expedient of reversing the action of its mutilated feet-rows, so as to bring the globe again upon its ab-oral pole and get its unmutilated feet-rows into action. At first sight, therefore, this result seems to point to the conclusion that the righting movements are of a merely serial kind; it seems to indicate that the feet are only able to act in one direction, from ab-oral to oral pole, and that there is not sufficient central co-ordination to induce them to act in the opposite direction, when it is found to be useless, from the interruption of the series, to continue the manœuvre in the ordinary direction. But a little closer thought will show that this conclusion is not justified by the facts. For even if we assume that the righting movements of the feet are entirely due to some central co-ordinating influence, it does not follow, when the execution of these movements is interrupted by the highly artificial means of shaving off one-half the feet-rows, that the central co-ordinating apparatus should be adapted to meet so unnatural a state of things.

Suppose, for instance, that it is an incipient sense of gravity that determines this central apparatus to work the feet-rows serially, in order to rotate the animal into its normal position; it does not follow that, under any circumstances, the stimulus supplied by this sense of gravity should induce the central apparatus to *reverse* the action of the feet-rows; for to do this would, under any circumstances, be to act in opposition to the stimulus supposed. Only if we were to imagine that the central apparatus, if present, must possess a true psychological element capable of sufficient intelligence to reflect that by temporarily acting in opposition to the sense of gravity the peculiar exigencies of the situation might be overcome—only then could we fairly argue that the result of these experiments shows the righting movements of the feet to be purely serial, or wholly independent of nervous co-ordination. As a matter of physiology, therefore, the only question which in the present connexion we have to consider is this —is the mechanism of the ambulacral feet so constructed as to insure that their serial action shall always take place in the same direction? For if it can be shown that their serial action may take place indifferently in either direction, it would follow that the persistency with which the shaved *Echini* continued reared upon their equators is the expression of some stimulus (such as a sense of gravity) continuously acting upon some central apparatus, and so impelling the latter to a continuous, though fruitless, endeavour at co-ordinating the absent feet. If the righting movements were wholly independent of any such central apparatus, and due only to the serial action of the feet, we should expect that (supposing the feet to be able to act serially in either direction) when the equator position had been attained in shaved specimens, it would not be maintained. For if there were no constant stimulus emanating from any co-ordinating centre persistently trying to induce the absent feet to continue the serial action in the same direction, we should expect, if serial action can take place in either direction, that after a time it should begin to take place in the opposite direction; upon the supposition that the feet may act serially in either direction, there is no more reason why a shaved *Echinus* should remain permanently reared upon its equator than there is that it should remain permanently inverted upon its pole, and therefore the fact that in the latter position the feet set about an immediate rotation of the animal, while in the former and quite as unnatural position they hold the animal in persistent stasis—this fact tends to show that the righting movements of the feet are something more than serial. The question, therefore, that we set ourselves to determine was, whether the serial action of the feet invariably takes place in the direction of ab-oral to oral pole, or may likewise take place in the opposite direction. We found that it may take place in the opposite direction, as the following observations prove. We have seen a shaved specimen, which after remaining for several hours on its equator was accidentally rolled over into its normal position, forthwith begin to rear itself upon its uninjured feet-rows. Executing this what we may call an inverted righting movement with activity, the *Echinus* was speedily reared into the equatorial position on the opposite side to that from which it had just fallen—and in order to do this, it

is needless to say, the feet of the uninjured rows had to be used serially in the direction opposite to that in which they are required to act when executing the ordinary righting manœuvre. We may wonder what the stimulus can have been which induced this *Echinus* spontaneously to rise upon its equator; but it is of interest in this connexion to add that, so soon as the equator position had been attained, and so soon therefore as any further action of the uninjured feet-rows in the same direction would have begun to get the animal into a position of ever-increasing difficulty as regards subsequent righting, so soon did the serial action in this direction cease, became reversed, and so again brought the animal gently into its normal position.

We have also seen wholly uninjured specimens when reaching the surface of the water by crawling up the sides of a tank, spontaneously rear themselves upon their equators and remain in that position for several minutes; but we have never observed a case of such rotation carried further than the equatorial line. The fact, however, that such rotation from oral to ab-oral pole can take place over half the whole length of a pair of feet-rows, proves that the feet may act serially in either direction. The same thing is further proved by the fact that single detached rays of Star-fish sometimes crawl backwards, and that in entire Star-fish the rays opposite to the direction of advance work their ambulacral feet centripetally, while those on the rays facing that direction work centrifugally.

Lastly, as proof that the ambulacral feet of *Echinus* are under the control of some centralising apparatus when executing the righting manœuvre, we may state one other fact. When the righting manœuvre is nearly completed by the rows engaged in executing it, the lower feet in the other rows become strongly protruded and curved downwards, in anticipation of shortly coming into contact with the floor of the tank when the righting manœuvre shall have been completed (see Plate 83, fig. 26). This fact tends to show that all the ambulacral feet of the animal are, like all the spines, held in mutual communication with one another by some centralising mechanism.

Such, then, is the evidence we have to adduce for the purpose of showing that the action of the ambulacral feet is not entirely or only of a serial kind, but is, in part at all events, dependent upon some centralising influence by which all the feet, like all the spines, are rendered capable of truly co-ordinated action. We have next to adduce our evidence to show that the action of the ambulacral feet, although as we have seen in some measure, is not exclusively dependent on this centralising influence.

(E.) In order to show this we must first narrate the experiments whereby we succeeded in ascertaining the central apparatus, on the integrity of which both the feet and the spines depend for their co-ordination. Having obtained the definite evidence of co-ordination which has now been fully detailed, we of course sought to localise the centre to which this co-ordination is due; and in searching for this centre our thoughts naturally turned to the only part of the nervous system where we could reasonably expect to find it. This part is the central nerve-ring, and, as we had anticipated,

experiment revealed unmistakable evidence of this being the centre of which we were in search.

If a circular incision be made all the way round the lantern of an *Echinus*, at a sufficient distance from the lantern to insure that the connexions of the nerve-ring with the rest of the organism shall be severed, the following results are produced :—

1. *Pedicels.*—Spontaneity impaired, though not destroyed. They are protruded, but not in such numbers or with so much activity as in the unmutilated animal; they, however, form their adhesions in the ordinary manner whenever they come into contact with a solid surface, and therefore their function of anchoring the *Echinus* securely remains unimpaired. They also still continue able to crawl, but they do so feebly and no longer in a determinate direction; the animal therefore advances slowly and in a very uncertain manner, frequently changing its direction of advance, and manifesting a marked tendency to rotate upon its own axis, either without moving from one spot or gyrating round and round some one or more centres in a wholly aimless way. The animals, however, are still able to climb perpendicular surfaces, though in a most uncertain manner. When stimulated strongly the activity of the animal is increased, but its power of escaping from the source of injury is completely destroyed; it crawls indifferently in any direction—as likely as not *towards* the source of injury—rotates upon its axis, and after crawling some distance in one direction may very likely reverse that direction, and so return to the place from which it started. All these movements, standing in such marked contrast to those exhibited by unmutilated specimens under similar circumstances, prove that the co-ordination of the ambulacral feet has been destroyed. On the other hand, the fact that they continue able to act *at all* proves that their activity is not *wholly* dependent upon the nerve-centre; all that the destruction of this centre entails is the destruction of their power of *co-ordinated* action.

When perfectly fresh and vigorous specimens are inverted, a proportion of about three to four remain permanently inverted till they die. As this is never the case with perfectly fresh and vigorous specimens when unmutilated, there can be no question that destruction of the nerve-centre exerts a profound influence on the action of the ambulacral feet upon which the execution of the righting manœuvre depends. On the other hand, the fact that a certain proportion of individuals continue able to execute this manœuvre after destruction of the nerve-centre—although they never do so without much difficulty and great expenditure of time—proves that the integrity of this centre is not absolutely essential to the execution of this manœuvre. Therefore, as experiment has failed to reveal to us any other general nerve-centre in the animal, and as even a segment of the animal containing but a single row of feet is in many cases able to perform this manœuvre, we conclude, as previously stated, that the action of the feet in performing these righting movements is partly of a serial character, although, for reasons mentioned in the two previous paragraphs, we further conclude that in the unmutilated animal these movements are largely assisted by the co-ordinating influence that emanates from the nervous centre.

2. *Pedicellariæ.*—No effect whatever is produced upon these organs by destruction of the nerve-ring.

3. *Spines.*—These organs, on the other hand, are profoundly affected—not, indeed, as regards their spontaneity and the function which they share with the pedicellariæ of closing round any instrument of stimulation, but as regards their other two more general functions. That the particular or local function which they share with the pedicellariæ should not be impaired by destruction of the general nerve-centre is no more than we might expect from those experiments detailed in previous parts of this paper, which proved that this function is performed exclusively by the numberless local nerve-centres (cells) of the external plexus. Thus, for instance, it will be remembered that when a small piece is cut out of the shell of an *Echinus* or *Spatangus*, and the internal surface of that piece painted with acid, its spines and pedicellariæ, although severed from any possible nervous connexion save those of the external plexus, will continue to perform their function of localising a seat of stimulation.

As regards, then, the more general function of the spines, we have first to consider what we may term their general reflex irritability—*i.e.*, their power of active bristling response all over the animal when any part of its surface is strongly stimulated, as by burning. Immediately after the operation of removing the nerve-centre this function is found to be in abeyance, or nearly so—strong stimulation of one part of the animal not being followed by any response of the spines in other parts. This effect, however, completely passes off within several hours after the operation, and is therefore to be attributed to shock. The fact, however, that the influence of shock is thus revealed in temporarily suspending this general nervous communication among the spines, proves that this general communication, unlike the more special one which they share with the pedicellariæ, is itself in communication with the central nervous ring. Further, the experiments detailed in a previous part of this paper prove that the medium of communication is in this case the hypothetical *internal* nervous plexus, as in the case just mentioned the medium of communication has been proved to be the *external* nervous plexus. And as the effect of the operation in question is only transitory —after recovery from shock the spines being as responsive as ever to severe stimulation—we must conclude that the general communication between the spines is maintained by the direct conductility of the supposed internal plexus, and is not of the nature of a reflex in which the nerve-ring is concerned as a general centre. The only effect of removing this nerve-ring is temporarily to paralyse, through shock, the supposed internal plexus with which the ring is connected.

Lastly, the effect of removing the nerve-ring is that of completely and permanently destroying the general co-ordination of the spines; that is to say, after this operation these organs are never again of any use to the *Echinus* for the purpose of locomotion. When the animal is placed upon a table and a lighted spirit lamp held against one side, although all the spines will manifest their active bristling movements, they will

not co-operate to move the animal away from the source of irritation, as is so invariably the case with unmutilated specimens. Removal of the nerve-ring has entirely destroyed the general co-ordination of the spines.

GENERAL SUMMARY.

I. Morphology.

In *Holothuria* the polian vesicle opens freely into a wide circular canal a short distance from the termination of the stone canal. From this circular canal five lozenge-shaped sinuses project forwards, and from each of these two large oval sinuses run forwards parallel with each other—the ten oval sinuses becoming continuous with the hollow stems of the tentacles. Injection of the polian vesicle shows that it forms one continuous tube system with the circular canal and its sinuses, oval sinuses and tentacles, pedicels and ampullæ. Unless the pressure is kept up for a considerable time there is no penetration of the injected fluid into the stone canal, and either the ring, vesicle, or sinuses, give way before the fluid reaches the madreporic plate. Specimens injected with a gelatine mass show that each canal sinus opens into a cæcal tube, which runs forwards internal to the sinuses of the tentacles as far as a wide circum-oral space. This space communicates by well-defined apertures with that portion of the body cavity which lies between the sinuses and the œsophagus, and which is reached through the circular apertures between the sinuses of the circular canal. Each canal sinus has three other apertures in its walls. It opens by a small round aperture into a radial canal, and the two other apertures occur as minute slits, one at each side of the orifice of the radial canal, leading into the adjacent tentacle sinuses. When the tentacle into which the sinus opens is protruded, there is no constriction between the sinus and the tentacle; but when the tentacle is retracted, there is a well-marked constriction at the junction of the sinus with the tentacle. The eversion of the perisome and the protrusion of the tentacles are effected chiefly by the shortening of the polian vesicle and the constriction of the longitudinal muscular bands, which run from the inner surface of the body-wall between each two adjacent tentacle-sinuses; but the circular fibres of the body-wall also assist in the process by contracting immediately behind the group of sinuses, so as to act on them by direct pressure, and also indirectly by forcing the body fluid against them.

The amount of the body-cavity fluid is constantly changing. At the entrance to cloacal chamber there is a circular valve which is constantly dilating and contracting, except when the aboral end of the animal is forcibly retracted. When open, this valve allows water to pass into the respiratory tree; when it begins to retract, water escapes from the cloaca. This alternate opening and closing takes place with perfect rhythm, at a rate of about six revolutions per minute. At the end of every seventh or eighth revolution a large stream of clear water is ejected, which sometimes contains

sand and the remains of food particles. When the tentacles are being protruded more water is taken in at the cloaca than escapes; on the other hand, retraction of the tentacles is preceded by an escape of a large stream of water.

In *Echinus* two tubes spring from the under surface of the madreporic plate. The one is dilated at its origin so as to include the greater portion of the plate, and ends in the so-called heart; the other is small, deeply pigmented, and runs along a groove in the heart to open into a circular canal at the base of the lantern. From the under aspect of this circular canal the five radial ambulacral vessels take their origin. Immediately within the oral margin of the shell and alternating with the inner row of pedicels, are the five pair of "tree-like organs." If a fine glass canula be forced through the membrane which extends from the apex of each tooth to the oral margin of the inter-ambulacral plates and sides of the alveoli, coloured fluids may be injected into the space between the membrane and the alveoli of the lantern; the fluid then slowly diffuses upwards into the vesicles around the apices of the teeth. It reaches these vesicles partly by passing directly upwards external to the alveoli, and partly by passing into the cavities of the alveoli and ascending through the circular sinus.

In *Spatangus* the ambulacral circum-oral canal has no polian vesicles or sinuses developed in connexion with it. Some of the pedicels have suckers, others are conical and devoid of them, while others again are flattened at their tips, and sometimes split up into segments.

If one of the arms of *Solaster papposa* is divided transversely and a coloured fluid is introduced into the open end of the radial canal, the ampullæ and pedicels of the injected arm are at once distended. The fluid next penetrates the circular canal, polian vesicles, ampullæ and pedicels of the other arms; but unless considerable pressure be kept up for some time, none of the solution enters the madreporic canal. If, however, the pressure is maintained for several hours with a column of fluid 2 feet high, the fluid ascends through the stone canal and diffuses slowly through the madreporic plate. When a thin slice is then shaved off the plate, the fluid is observed escaping from a small circumscribed area situated between the centre and the margin of the plate, and corresponding in size and position with the termination of the stone canal on the inner surface. The stone canal gradually increasing in diameter as it passes inwards from the madreporic plate, runs obliquely over its accompanying sinus, till it finally hooks round this sinus to open into the circular canal. Springing from this canal and opposite to each inter-radial space (with the exception of the space occupied by the stone canal) is a polian vesicle. The size and form of these vesicles are largely determined by the amount of fluid in the pedicels. In none of the injected specimens was there any evidence of a communication between the ambulacral vessels and the body cavity, or between the ambulacral and the blood (neural) vessels. There was, however, abundant evidence of communication between the latter and the exterior. When a canula was introduced into the outer end of the sinus, a coloured solution could be easily forced through the sinus into the circular

blood-vessel, and from this into the radial blood-vessels. But when the canula was introduced into the proximal end of the sinus, the solution rapidly rushed along the sinus and escaped through the madreporic plate—proving that the blood-vessels of *Solaster* communicate far more freely with the exterior than do the water-vessels.

The ambulacral system of the common Star-fish only differs from that of the Sun-star in having no polian vesicles. *Astropecten*, on the other hand, has polian vesicles; but in it the pedicels have departed from the usual form in being short, conical, and unprovided with terminal suckers. In *Ophiura* the pedicels are morphologically similar to those of *Astropecten*, though shorter and more slender. They diminish in size as they proceed outwards, and at the ends of the arms are scarcely visible.

II. Physiology.

1. *Natural movements.*—The ordinary crawling movements of *Astropecten aurantiacus* are peculiar, the ambulacral feet acting the part of walking poles and cilia combined. Brittle-stars progress by using two opposite arms upon the floor of the tank, with a sort of leap, and can thus travel at the rate of 6 feet per minute. The ordinary progression of *Echinus* and *Spatangus* is assisted by the co-ordinated action of the spines, and when placed upon a flat surface out of the water the animal advances by means of its spines alone. In *Echinus* the lantern and pedicellariæ are also used to assist in locomotion.

All the Echinodermata that we have observed are able, when placed upon their dorsal surfaces on the floor of a tank, to recover their normal position on their oral surface. The common Star-fish does so by twisting the ends of two or more of its rays round, so as to bring its terminal suckers into action upon the floor of the tank, and then, by a successive and similar action of the suckers further back in the series, the whole ray is progressively twisted round, so that its ambulacral surface is applied flat against the floor. The rays which perform this action twist their semi-spirals in the same direction, and by their concerted action serve to drag the disc and the remaining rays over themselves as a fulcrum. Other species of Star-fish, which have not their ambulacral suckers sufficiently developed to act in this way, execute their righting movements by doubling under two or three of their adjacent rays, and turning a somersault over them, as in the previous case. *Echinus* rights itself when placed on its ab-oral pole, by the successive action of two or three adjacent rows of suckers—so gradually rising from ab-oral pole to equator, and then as gradually falling from equator to oral pole. *Spatangus* executes a similar manœuvre entirely by the successive pushing and propping action of its longer spines.

2. *Stimulation.*—All the Echinoderms that we have observed seek to escape from injury in a direct line from the source of stimulation. If two points of the surface are stimulated, the direction of escape is the diagonal between them. When several points all round the animal are simultaneously stimulated, the direction of advance

becomes uncertain, with a marked tendency to rotation upon the vertical axis. If a short interval of time be allowed to elapse between the application of two successive stimuli, the direction of advance will be in a straight line from the stimulus applied latest. If a circular band of injury be quickly made all the way round the equator of *Echinus*, the animal crawls away from the broadest part of the band—*i.e.*, from the *greatest amount* of injury.

The external nerve-plexus supplies innervation to three sets of organs—the pedicels, the spines, and the pedicellariæ; for when any part of the external surface of *Echinus* is touched, all the pedicels, spines, and pedicellariæ within reach of the point that is touched immediately approximate and close in upon the point, so holding fast to whatever body may be used as the instrument of stimulation. In executing this combined movement the pedicellariæ are the most active, the spines somewhat slower, and the pedicels very much slower. If the shape of the stimulating body admits of it, the forceps of the pedicellariæ seize the body and hold it till the spines and pedicels come up to assist.

And here we have proof of the function of the pedicellariæ. In climbing perpendicular or inclined surfaces of rock covered with waving sea-weeds, it must be no small advantage to an *Echinus* to be provided on all sides with a multitude of forceps adapted, as described, to the instantaneous grasping and arresting of a passing frond. For in this way not only is an immediate hold obtained, but a moving piece of seaweed is held steady, till the pedicels have time to establish a further and more permanent hold upon it with their sucking discs. That this is the chief function of the pedicellariæ is indicated by the facts that, 1st, if a piece of sea-weed is drawn over the surface of an *Echinus*, this function may clearly be seen to be performed; 2nd, that the wonderfully tenacious grasp of the forceps is timed as to its duration with an apparent reference to the requirements of the pedicels, for after lasting about two minutes (which is about the time required for the suckers to bend over and fix themselves to the object held by the pedicellariæ if such should be a suitable one) this wonderfully tenacious grasp is spontaneously released; and 3rd, that the most excitable part of the trident pedicellariæ is the inner surface of the mandibles, about a third of the way down their serrated edges—*i.e.*, the part which a moving body cannot touch without being well within the grasp of the forceps. When the forceps are closed, they may generally be made immediately to expand by gently stroking the external surface of their bases.

With regard to stimulation of the spines, if severe irritation be applied to any part of the external or internal surface of an *Echinus*, the spines all over the animal take on an active bristling movement. The tubercles at the bases of the spines are the most irritable points on the external surface.

With regard to stimulation of the pedicels, if an irritant be applied to any part of a row, all the pedicels in that row retract in succession from the seat of stimulation, but the influence does not extend to other rows. A contrary effect is produced by

applying an irritant to any part of the external nerve-plexus, all the pedicels being then stimulated into increased activity. Of these antagonistic influences the former, or inhibitory one, is the stronger; for if they are both in operation at the same time, the pedicels are retracted.

Star-fish (with the exception of Brittle-stars) and *Echini* crawl towards, and remain in, the light; but when their eye-spots are removed they no longer do so. When their eye-spots are left intact they can distinguish light of very feeble intensity.

3. *Section* (A.).—*Star-fish.*—Single rays detached from the organism crawl as fast and in as determinate a direction as do entire animals. They also crawl towards light, away from injuries, up perpendicular surfaces, and when inverted right themselves. Dividing the ray-nerve in any part of its length has the effect of destroying all physiological continuity between the pedicels on either side of the division. Severing the nerve at the origin of each ray, or severing the nerve-ring between each ray, has the effect of totally destroying all co-ordination among the rays; therefore the animal can no longer crawl away from injuries, and when inverted it forms no definite plan for righting itself—each ray acting for itself without reference to the others, there is, as a result, a promiscuous distribution of spirals and doublings, which as often as not are acting in antagonism to one another. This division of the nerves usually induces, for some time after the operation, more or less tetanic-like rigidity of the rays. The operation, however, although so completely destroying physiological continuity in the rows of pedicels and muscular system of the rays, does not destroy, or perceptibly impair, physiological continuity in the external nerve-plexus; for however much the nerve-ring and nerve-trunks may be injured, stimulation of the dorsal surface of the animal throws all the pedicels and muscular system of the rays into active movement. This fact proves that the pedicels and muscles are all held in nervous connexion with one another by the external plexus, without reference to the integrity of the main trunks.

(B.) *Echini.*—If a cork-borer be rotated against the external surface of an *Echinus* till the calcareous substance of the shell is reached, and therefore a continuous circular section of the overlying tissues effected, the spines and pedicellariæ within the circular area are physiologically separated from those without it, as regards their local reflex irritability. That is to say, if any part of this circular area is stimulated, all the spines and pedicellariæ within that area immediately respond to the stimulation in the ordinary way, while none of the spines or pedicellariæ surrounding the area are affected, and conversely. Therefore we conclude that the function of the spines and pedicellariæ of localising and gathering round a seat of stimulation, is exclusively dependent upon the external nervous plexus. If the line of injury is not a closed curve, so as not to produce a physiological island, the stimulating influence will radiate in straight lines from its source, but will not irradiate round the ends of the curve or line of injury.

Although the nervous connexions on which the spines and pedicellariæ depend for

their function of localising and closing round a seat of stimulation are thus shown to be completely destroyed by injury of the external plexus, other nervous connexions, upon which another function of the spines depends, are not in the smallest degree impaired by such injury. This other function is that which brings about the general co-ordinated action of all the spines for the purposes of locomotion. That this function is not impaired by injury of the external plexus is proved by severely stimulating an area within a closed line of injury on the surface of the shell; all the spines over the whole surface of the animal then manifest their bristling movements, and by their co-ordinated action move the animal in a straight line of escape from the source of irritation.

We have, therefore, to distinguish between what may be called the local reflex function of the spines, which they show in common with the pedicellariæ and which is exclusively dependent upon the external plexus, and what we may call the universal reflex function of the spines, which consists in their general co-ordinated action for the purposes of locomotion, and which is wholly independent of the external plexus. Apparently, therefore, this more universal function must depend upon some other set of nervous connexions (which, however, we have not been able to detect histologically), and experiment shows that these, if present, are distributed over all the *internal* surface of the shell. For if the internal surface be painted with acid, or scoured out with emery paper and brick-dust, the spines and pedicellariæ, after a short period of increased activity or bristling, become perfectly quiescent, lie flat, and lose both their spontaneity and irritability. After a few hours, however, the spontaneity and irritability of the spines return, though in a feeble degree, and also those of the pedicellariæ in a more marked degree. These effects take place over the whole external surface of the shell, if the whole of the internal surface be painted with acid or scoured with brick-dust; but if any part of the external surface be left unpainted or unscoured, the corresponding part of the external surface remains uninjured. From these experiments we conclude:—1st, that the general co-ordination of the spines is wholly dependent on the integrity of the hypothetical internal plexus; 2nd, that the hypothetical internal plexus is everywhere in intimate connexion with the external, apparently through the calcareous substance of the shell; and 3rd, that complete destruction of the former, while profoundly influencing through shock the functions of the latter, nevertheless does not wholly destroy them.

Echini may be divided into pieces, and the pedicels, spines, and pedicellariæ upon these pieces will continue to exhibit their functions of local reflex irritability, however small the pieces may be. If an entire double row of pedicels be divided out as a segment, and then placed upon its ab-oral end, it may rear itself up on its oral end by the successive action of its pedicels, and then proceed to crawl about the floor of the tank. We have therefore to meet the question:—Is the action of the ambulacral feet in executing these righting movements of a merely serial kind—A, B, and C, first securing their hold of the tank floor, owing to the stimulus supplied by contact, and

then by their traction tilting over the globe, till D, E, and F, are able to touch the floor, and so on; or does the righting action depend upon nervous co-ordination? We conclude that both principles are combined—the action of the pedicels being serial, but also assisted by nervous co-ordination. This conclusion is sustained, not only by the movements of an unmutilated *Echinus* when suspended by a thread against the wall of a tank, but also by the experiment of shaving off the spines and pedicels over one half of one hemisphere—*i.e.*, the half from the equator to the oral pole. When then inverted and forced to use their mutilated pedicel-rows, the *Echini* reared themselves upon their equators, and then, having no more pedicels wherewith to continue the manœuvre, came to rest. This rest was permanent—the animal remaining, if accidents were excluded, upon its equator till it died. The question, then, here seems to resolve itself simply into this:—Is the mechanism of the pedicels so constructed as to ensure that their serial action shall always take place in the same direction; for if it can be shown that their serial action may take place indifferently in either direction, it would follow that the persistency with which the partly-shaved *Echini* continue reared upon their equators, is the expression of some stimulus (such as a sense of gravity) continuously acting upon some central apparatus, and impelling the latter to a continuous, though fruitless, endeavour to co-ordinate the absent pedicels. If the pedicels are able to act serially in either direction, there is no more reason why a partly-shaved *Echinus* should remain permanently reared upon its equator, than that it should remain permanently inverted upon its pole; and therefore the fact that in the latter position the pedicels set about an immediate rotation of the animal, while in the former and quite as unnatural position they hold the animal in persistent stasis—this fact tends to show that the righting movements of the pedicels are something more than serial. Thus the whole question as between the two hypotheses amounts to whether the pedicels are able to act serially from oral to ab-oral pole. Observation has shown us that they are so, for we have seen *Echini* spontaneously rear themselves from their normal position on the oral pole, to the position of resting upon their equators. Further, as additional evidence that the righting movements are at least assisted by some centralizing influence, is the fact that when the evolution is nearly completed by the pedicel-rows engaged in executing it, the lower pedicels in the other rows become strongly protruded and curved downwards, in anticipation of shortly coming into contact with the floor of the tank.

But, on the other hand, there is evidence to show that the action of the pedicels in executing this manœuvre, although as we have seen in some measure, is not exclusively dependent upon this centralizing influence. We found that the centre from which this influence proceeds is the nerve-ring that surrounds the lantern. For when this is removed, the following results are produced: the pedicels have their spontaneity impaired, though not destroyed—the animal still continuing to crawl, but only feebly, and no longer in a determinate manner, frequently changing its direction of advance, and showing a marked tendency to rotate upon its vertical axis. Moreover, the

Echinus is now no longer able to escape from injury, but when stimulated crawls indifferently in any direction. Thus, removal of the nerve-centre seriously impairs the activity of the pedicels, and totally destroys their co-ordination. Yet when specimens so mutilated are inverted, one out of every four specimens is able to right itself. This, however, is only done with great difficulty and after a long time, so that, under these circumstances, the execution of this manœuvre seems to be just barely possible. Still the fact of its being possible at all proves that the integrity of the nerve-centre is not absolutely essential to its performance. Therefore, as experiment has failed to reveal to us any other general nerve-centre in the animal, and as even a segment of the animal containing only a single row of pedicels is in many cases able to perform this manœuvre, we conclude, as already stated, that the action of the pedicels is partly of a serial character, though largely assisted by the co-ordinating influence that emanates from the nerve-centre.

The effect of this operation upon the spines and pedicellariæ still remains to be considered. No effect at all is produced upon the pedicellariæ; but upon the spines a profound influence is seen to be exercised. Their spontaneity, indeed, remains unimpaired, as does also the function which they share with the pedicellariæ of closing round any instrument of stimulation; likewise their power of responsive bristling all over the animal when any part of the animal is severely stimulated continues to be manifested as before, although for an hour or two after the operation this power is suspended by shock. But the general *co-ordination* of the spines is totally and permanently destroyed; for if the animal be placed upon a table and a spirit lamp flame held against one side, although the spines will manifest their bristling movements (if the period of shock has been allowed to pass away), they will no longer co-operate to remove the animal from the source of irritation. These facts prove, 1st, that the general co-ordination of the spines is wholly dependent upon the nerve-centre; 2nd, that the spontaneity and local reflex irritability are wholly independent of that centre—they depend entirely upon the external nerve-plexus; and 3rd, that the universal nervous connexions revealed in the bristling movements of the spines, and which as shown by previously narrated experiments depend upon the hypothetical *internal* nerve-plexus, are themselves in nervous connexion with the nerve-centre. For only thus can we explain the long period of shock which removal of this centre entails upon the functions of this supposed internal plexus. Nevertheless, the fact that these functions are eventually resumed in the general bristling of the spines, proves that this general communication between the spines is maintained by the direct conductibility of the supposed internal plexus, and is not of the nature of a reflex in which the nerve-ring is concerned as a general centre for the responsive, as distinguished from the co-ordinated, action of the spines.

Literature.

For a general account of the literature on the morphology of the Echinoderms up to 1872 reference had best be made to Baudelot in 'Archives de Zoologie Exp. et Gen.,' t. i., pp. 176–216.

The observer who first detected the nervous ring in *Echinus* was Van Beneden, who published his description in 'L'Echo du Monde Savant' in 1835. A more detailed description was afterwards given by Krohn in Müller's 'Archives' in 1841, which was followed by the well-known investigations of Valentin*, L. and A. Agassiz†, J. Müller‡, Hoffmann§, and Lovén‖. More recently still a valuable paper has been published by Fredericq¶, of the existence of which we were not aware until our own paper had been written. As we now find that some of our results have been anticipated by this author, we shall here devote a few paragraphs to epitomising all the more important features of his work.

Fredericq found that the pentagonal nerve-ring of *Echinus* and its five radial nerves are all contained in as many sheaths or tubes of membrane, which are mesentery-like expansions of the lining membrane of the shell. These enveloping tubes send out lateral branches which contain the lateral nervous offshoots; the latter pass out of the ambulacral pores in company with the pedicels which they serve to enervate, a delicate nerve running along the whole length of each pedicel to terminate at its distal end in a tactile organ.

Fredericq considers it probable that in their passage through the ambulacral pores the nerves also send branches to the spines and pedicellariæ; these branches, however, he failed to detect. The radial or ambulacral nerve-trunks terminate in the ocular plates. The latter, however, show no histological evidence of supporting any structure resembling an ocular apparatus; and Fredericq could obtain no physiological evidence of sensibility either to solar or to artificial light. He does not state clearly what his experiments in this connexion were, and so we infer that they cannot have been the same as ours. He regards the pigment spot as a "fiction."

The nerve-ring sends off, in addition to the ambulacral trunks, the nerve-cords to the intestine. In the ring and trunks there is no differentiation into ganglia and fibres, but the whole is in structure uniform and in function central. The brown colour is due to elongated and irregular cells having conspicuous nuclei filled with pigment, and supposed to betoken connective tissue. The nervous tracts are them-

* 'Anatomie du genre *Echinus*,' 1841.

† 'Bulletin of the Museum of Comp. Zool.,' in vols. i., ii., and iii., "Revision of the *Echini*;" and 'American Naturalist,' vol. vii., pp. 398–406.

‡ 'Abhandl. der Konigl. Akad. der Wissens. zu Berlin,' 1853, and Müller's 'Archiv.,' 1853, p. 175, and 1850, p. 127.

§ 'Nederländisches 'Archiv. für Zool.,' i., 1871, p. 54.

‖ 'Anns. and Mag. Nat. Hist.,' 1872, p. 28, and 'Études sur les Échinoïdées,' 1875.

¶ 'Arch. de Zool. Exp.,' t. v., pp. 429–440.

selves composed of two distinct layers—an external and an internal, each of which presents a uniform structure of cells and fibres.

In a division of his paper devoted to an account of physiological experiments, FREDERICQ records several observations which are identical with some of those recorded in the corresponding division of the present paper. Thus, he tried the effect of severing the five ambulacral trunks, and found, as we found, that, while the operation did not entail paralysis of the pedicels, it did entail complete destruction of co-ordination between their five rows. He also observed the righting movements of inverted *Echini*, and experimented on the effect upon these movements of severing the nerve-ring. Here, however, his results are not quite in accordance with ours, for he says that in no case does an *Echinus* when so mutilated succeed in righting itself; whereas we found, as before stated, that out of twelve perfectly fresh specimens so mutilated three were able to right themselves. We are quite sure that our results in this connexion are trustworthy; for, as these results were contrary to our expectations, we took the precaution of altogether removing the lanterns from a number of perfectly fresh specimens, and found, as previously, that a proportion of one in four continued able to right themselves.

Lastly, FREDERICQ observed the local reflex action of the spines and pedicellariæ, and also the insulating effect upon this action of a closed line of section. He inferred from these observations the presence of an external plexus, but was unable to detect its presence histologically.

We may now conclude this account of previous literature with a discussion of previous theories as to the function of the pedicellariæ.

In stating our opinion as to what we consider their main function, it seems desirable briefly to consider the functions which have been ascribed to these organs by previous observers. Professor OWEN supposes, or supposed ('Comp. Anat. of Inverts,' p. 203), that their work in the economy of the animal is that of removing parasitic growths from the shell; and somewhat allied to this view is that of Professor A. AGASSIZ, who regards the function of these appendages to be that of "scavengers." He says: "If we watch a Sea Urchin after he has been feeding, we shall learn at least one of the offices which this singular organ performs in the general economy of the animal. That part of the food which he ejects passes out of the anus—an opening on the summit of the body in the small area where the zones of which the shell is composed converge. The rejected particles, thrown out in the shape of pellets, are received on their little feeler-like forceps, and they are passed from one to the other down the side of the body till they are dropped off into the water. Nothing is more curious and entertaining than to watch the neatness and accuracy with which this process is performed. One may see the rejected bits of food passing rapidly along the lines upon which the pedicellariæ occur in greatest numbers, as if they were so many little roads for the carrying away of the refuse matter. Nor do the forks cease from their labour till the surface of the animal is completely clean and free from any foreign substance. Were

it not for the pedicellariæ the fæces thus rejected would be entangled among the tentacles and spines, and remain stranded there till the motion of the water washed it away. . . . These curious little organs have other offices besides this very laudable and useful one of scavengers: they occur over the whole body, while they pass the excrements only along certain given lines. They are especially numerous about the mouth, where they are much shorter and more compact. The muscular sheath below the head is quite short, the tripartite head resting directly upon the lines of the base. On watching the movements of the pedicellariæ we find that they are extremely active, opening and shutting their forks unceasingly, reaching forward in every possible direction, the flexibility of the sheath enabling them to sweep in all the corners and recesses between the spines; and occasionally they are rewarded by catching hold of some unfortunate little crustacean, worm, or mollusca which has become entangled among the spines. They do not seem to pass their prey to the mouth—at least, I have never succeeded in seeing Sea Urchins pass the food thus caught—but merely threw it off from the surface like any other refuse matter. Their mode of eating, also—a sort of browsing, by means of their sharp teeth, along the surface of the rocks—does not favour the idea of using the pedicellariæ as feelers."*

From this account we gather that Professor AGASSIZ regards the main function of the pedicellariæ to be that of removing excrement, although they may also act the part of general cleansers to remove any other undesirable substance—whether parasitic or otherwise—from the general surface. This view has recently been confirmed by Mr. W. O. SLADEN, who says, in the course of an interesting paper,† "Mr. ALEXANDER AGASSIZ was, I believe, the first, who by actual observation assigned the true function to any of these organs. Unfortunately, Mr. AGASSIZ leaves the matter without saying which of the forms of this appendage was the agent employed. I also have seen the same operation performed; and it was always the *pedicellariæ tridentes* that came into use for the purpose; indeed, the most superficial examination would suggest that these alone could be employed for such a service, neither the *pedicellariæ globiformes* nor the *pedicellariæ triphyllæ* having valves capable of grasping so large a body as the ejected pellets in question. On the other hand, the jaws of the *pedicellariæ tridentes* are admirably fitted for the purpose; and that this is the chief use of that form of pedicellariæ there seems but little doubt."

Now, restricting our consideration in the first instance to this form of pedicellaria, it seems *à priori* improbable that so elaborate and peculiar a structure should have been developed for the purpose supposed. To sustain the supposition, it would at least require to be shown that the excrement of *Echinus* is so difficult of removal that were it not for the action of these pedicellariæ it would remain on the ab-oral pole of the animal in an amount, or for a time, that might be injurious. Yet we can confidently say that such is not the case. The dung-pellets are *generally* removed by the action

* 'American Naturalist,' vol. xii., pp. 399–400, 1873.

† 'Annals and Magazine of Nat. Hist.,' Aug., 1880, p. 101, *et seq.*

of the *spines*, without assistance from any of the pedicellariæ. Being of a nearly spherical shape and somewhat firm consistence, their tendency is to roll down the sides of the globular animal, and to facilitate this movement the spines in their course diverge as soon as contact takes place—the pellet being thus allowed to pass between the spines with very little delay, and when brought up by other spines or stalks of pedicellariæ lower down in the series, these similarly diverge, and so on, till the pellet arrives at the equator of the animal, when it drops perpendicularly off the shell. We have watched the process over and over again, and have been so struck with the methodical action of the spines concerned in it that we find it impossible to entertain any doubt on the question whether they are alone sufficient to perform the process, or require any considerable assistance from the pedicellariæ. The assistance which the latter organs furnish in this process is only occasional, and seldom seems to be urgently required; it appears to us clearly but an accessory, if not an accidental function, for the performance of which the development of these organs cannot have been necessitated. And, direct observation apart, this view would seem to be rendered sufficiently obvious by the fact that these pedicellariæ are not restricted to the upper hemisphere of the animal, but occur also below the equatorial line, where they can never have a chance of seizing a dung-pellet at all.

Direct observation again apart, the homologies of the pedicellariæ would alone suggest that their function is probably subservient to locomotion. The valuable paper of Professor Agassiz already quoted derives its value from the clear demonstration which it supplies that the pedicellariæ are modified spines, and therefore the most obvious view would seem to be that they have been modified for the purpose of acquiring special proficiency as grasping organs of locomotion, over and above that which is presented by the unmodified or stilt-like spine.

On the whole, then, we believe that at least the *pedicellariæ tridentes* are not only homologous with, but analogous to, the spines; and therefore we demur to the statement of Professor Agassiz when he says, "the same reasoning will readily suggest to the student of Echinoderms the homology of the so-called claws of Ophiurians and of the anchors of Holothurians which, *although used for such totally different functions*, being a sort of prehensile organ for locomotion along the ground, are in reality only in their turn modified spines, or different forms of pedicellariæ." There can be no question about the homology, and our observations have satisfied us that there can be as little question about the analogy. The opinion, therefore, which we have here italicised and which refers to the pedicellariæ of *Echinus*, we think requires amendment; for observation has shown us that these organs here perform the same kind of functions as those which Professor Agassiz recognises the homologous organs to perform in Ophiurians and Holothurians, where—as he says, in words which follow the above quotation—"the pedicellariæ hooks and anchors perform the part of organs of prehension and locomotion at the same time."

Thus far we have been considering the case of the *pedicellariæ tridentes*. The other

forms, having such much smaller grasping organs, might at first appear to be of no use in grasping sea-weeds. That they are not of so much use in this respect as the *tridentes* is rendered obvious by experiment; and therefore we do not doubt that these pedicellariæ are proportionally of more service than the *tridentes* as general cleansers. We cannot say that we have ourselves observed any actual evidence of such being the case; but the interesting experiments of Mr. SLADEN (which were not published till our observations had been completed) leads him to "offer it as a suggestion" that "the ciliary epithelium is altogether insufficient" to keep the general surface clean, and that "the duty devolves upon the *pedicellariæ globiferæ*, the following being the manner in which the work is performed:—

"When the tactile cushion of the pedicellariæ comes into contact with a tangible object of foreign matter, the valves close and a discharge of mucus takes place* wherewith the obnoxious object is covered. When the hold of the jaws is again relaxed, the irritating substance remains entangled in a cloud of the glairy exudation, ready to be easily disengaged from the surface of the animal by a few movements of the neighbouring spines, and is finally carried off by the ordinary currents of the water in which the *Echinus* lives.

"A similar process may be observed with the greatest ease to be carried out by *Astropecten;* and this I have been able to verify many times by placing a specimen of the common *A. aurantiacus* in a large flat vessel, convenient for observation, and when covered with sea-water sprinkled some fine sand and mud over its dorsal area. In the course of a short time most of this will have been carried away by the action of the paxillæ and by the lateral papillated grooves, whilst such particles as have resisted this operation will be found enveloped in a glairy pellicle, which is gradually and by very slight motion drawn into a narrow band extending over the median line of each ray. This is then disengaged from the surface entirely, and is finally cast off by the slightest movement the Star-fish may make."

POSTSCRIPT.

[Received December 2, 1881.]

Since this paper was sent in there has been a note published by MM. GEDDES et BEDDARD in 'Comptes Rendus' (tom. xcii., pp. 308–10) on the histology of the muscular tissues of Echinoderms. According to the observations of these authors the conflicting views of previous observers on the question whether the muscular fibres of Echinoderms are striated or unstriated, admit of being reconciled by the fact that the same muscle fibres are sometimes unstriated and at other times appear to be striated.

* Mr. SLADEN is here describing the functions of pedicellariæ globiferæ in a particular species of *Echinus* (*Sphærechinus granularis*) where such a discharge was found by him to be very copious.

Further, since this paper was sent in, our attention has been drawn by Dr. P. M. Duncan, and also by Professor Lovén himself, to an important paragraph in the work of the latter already referred to ('Études sur les Échinoïdées,' page 8). This paragraph, which had previously escaped our notice, is as follows:—

"Chacun des cinq grands troncs nerveux qui naissent des angles du collier, et qui parcourent la face interne des ambulacres, le long de leur suture médiane, fournit dans chaque plaque un ou deux nerfs, qui se dirigent chacun vers le pore tentaculaire correspondant. Conjointement avec le vaisseau aquifère du tentacule, le nerf s'y enfonce, pour reparaître sur la face externe au dehors de la couche calcaire du test. Sur ce trajet il doit fournir des filets nerveux au tentacule et au sphéricle, bien qu'on ne soit pas parvenu à en démontrer l'existence. Mais, comme ou peut le voir chez la Brissopsis lyrifera, et le plus distinctement sur la troisième plaque du bivium avoisinant le sternum et plus dépourvue de radioles que le autres, le nerf, en sortant du pore tentaculaire sur la face externe du test, se ramifie en un grand nombre du filets, lesquels, après avoir traversé la plaque diagonalement, se distribuent aux aires interradiales en formant des entrelacements serrés et riches en cellules ganglionnaires. On conçoit que, tous les rameaux du tronc nerveux se divisant de cette manière, il y aura, répandu à la surface du corps, un système nerveux périphérique extrêmement développé, fournissant des nerfs aux radioles, aux pedicellaires, aux clavules des fascoïles, et, en général, à toutes les parties externes. La figure donnée en représente une très petite partie dessinée à un fort grossissement."

It only remains to add that during the past autumn we have continued the research, and have been successful in obtaining full histological demonstration of the internal nervous plexus of *Echinus*. This plexus is therefore now no longer "hypothetical;" and in a subsequent paper its character, distribution, and mode of communicating with the external plexus will be fully described, together with some further physiological experiments.

G. J. R.,
J. C. E.

December 1, 1881.

Description of the Figures.

PLATES 79–85.

Fig. 1. Ambulacral system of *Holothuria*.

- *a*. Polian vesicle.
- *b*. Ambulacral circular canal.
- *c*. Circular canal sinus. Small circle at apex indicates position of opening into a radial canal (*k*); at each side the opening into a tentacle sinus (*d*) is indicated.
- *e*. Trunk of tentacle.

f. Aperture between sinuses which leads into space around œsophagus. This space communicates with circumoral sinus (Plate 79, fig. 2, *a*).
g. Ampulla.
h. Circle indicating constriction between oval sinus and retracted tentacle.
i. Madreporic plate.
k. Radial canal giving off two lateral branches, each branch gives rise to an ampulla (*g*) and a pedicel (*n*).
l. Longitudinal muscular band.
m. Retractor muscle.

Fig. 2. Section through anterior portion of *Holothuria*.
a. Circumoral sinus.
b. Circular ambulacral canal.
c. Sinus of circular canal.
d. Sinus of tentacle.
e. Aperture through which circumoral sinus communicates with space around œsophagus.
f. Cæcal prolongation from sinus of circular canal.
h. Fibres stretching inwards from sinuses and circular canal to œsophagus.
m. Retractor muscle.

Fig. 3. *Holothuria* with tentacles fully distended.
c. Circumoral sinus.
b. Aperture through which sinus communicates with space around œsophagus.
a. Cæcal end of prolongation from sinus of circular canal.
d. Aperture of cloacal chamber.

Fig. 4. Section of cloaca in *Holothuria*; valve open and retracted.
Fig. 5. Section of cloaca in *Holothuria*; valve closed and projected.
Fig. 6. Partially retracted pedicel of *Echinus* showing sucking disc.
Fig. 7. Section of partially retracted pedicel of *Astropecten*.
Fig. 8. Section of pedicel of *Astropecten* slightly retracted at one side (*a*) so as to act as a sucker.
Fig. 9. Section of pedicel of *Ophiura*.
Fig. 10. Section through lantern of *Echinus*.
a. Sinus under madreporic plate (*b*).
c. Madreporic canal.
d. "Heart."
f. Radius.
g. Rotula.
h. Circular muscle.
i. Radial canal.
j. One of the inner series of pedicels with a tubular ampulla (*k*).

l. "Tree-like organ."
m. Membranous wall of large space above "tree-like organ."
n. Sinus around tip of tooth
o. Tooth lying in space (*p*) between the alveoli.
q. Edge of alveolus.
r. Interalveolar muscle.
s. Auricle.
t. Radial nerve.

Fig. 11. Ambulacral system—*Solaster*.
a. Madreporic canal.
b. Inner end (*g*) outer end of sinus leading to circular neural vessel (*h*) from which radial neural vessels (*l*) arise.
c,*d*. Polian vesicles.
f. Ampullæ.
m. Oral aperture.
n. Madreporic plate.

Fig. 12. Madreporic plate enlarged, showing space (*a*) through which fluid escapes from madreporic canal.

Fig. 13. Scheme showing portions of ambulacral and nervous systems of *Echinus*.
a. Ampullæ.
b. Radial nerve.
c. Neural radial sinus.
d. Lining membrane of shell.
e. Radial ambulacral canal.
f. Lateral branch of radial canal.
g. Pedicel.
h. Spine.
i. Pedicellaria.
k. Layer of fibres external to shell.
l. Subepidermic nerve-plexus.
l'. Plexus extending over base of spine.
l''. Plexus extending over pedicellaria towards base of mandibles.
m. Epidermis.
n. Lateral branch from nerve-trunk.
o. Continuation of lateral branch alongside of pedicel.
p. Portion of lateral branch which probably communicates with external plexus.
r. Ambulacral plate.

Fig. 14. Fibres and cells of nerve-trunk.

Fig. 15. External plexus partially covered by epidermis lying over muscular and connective tissue-fibres.

Fig. 16. Nerve-cells lying amongst muscular fibres at base of spine.
Fig. 17. Nerve-fibres lying between epidermis and calcareous stem of pedicellaria.
Fig. 18. Plexus lying over muscular fibres near base of mandibles of pedicellaria.
Fig. 19. Natural co-ordinated movements of common Star-fish.
Fig. 20. Natural righting movements of common Star-fish.
Fig. 21. Natural righting movements of *Astropecten*.
Fig. 22. Natural locomotor movements of *Ophiura*.
Figs. 23–26. Natural righting movements of *Echinus*.
Fig. 27. Righting movements of severed ray of common Star-fish after bisection of radial nerve.
Fig. 28. The unco-ördinated movements of inverted Star-fish after section of the five radial nerves.
Fig. 29. Tulip-shape assumed by the same.
Figs. 30, 31. Righting movements of segments of *Echinus*.
Figs. 32–34. Righting movements of *Echinus* on a vertical surface.

XX. *On the Influence of the Atomic Grouping in the Molecules of Organic Bodies on their Absorption in the Infra-red Region of the Spectrum.*

By Captain ABNEY, *R.E., F.R.S., and Lieut.-Colonel* FESTING, *R.E.*

Received February 5—Read February 10, 1881.

[PLATES 86–88.]

Introductory.

THE researches on which this paper is founded were commenced in February, 1880, but were not sufficiently advanced for any communication to be made regarding them during last session. As an article on the absorption of colourless liquids by Dr. W. RUSSELL, F.R.S., and Mr. LEPRAIK appeared in 'Nature' on the 19th August, 1880, it might have been thought that we were merely following in the steps of those gentlemen, of the scope of whose work we were not aware; we have therefore stated when our work commenced. It will also be seen that our work has been more especially confined to the infra-red region, for reasons which will presently appear, whilst Messrs. RUSSELL and LEPRAIK turned their attention to the visible portion of the spectrum.

Reasons for undertaking the research.

The investigations of Professor TYNDALL on radiant energy, and its absorption by various organic compounds, led us to believe that if such marked effects were observed by means of the thermopile, at least as much information ought to be gathered from the photographic method recently brought to the notice of the Royal Society.* The absorption measured by the thermopile is essentially the integration of all the absorptions in the different regions of the spectrum examined, and by this method it is almost impossible to determine the position of the several components, since the face of the instrument has an appreciable breadth. By the photographic method not only can an approximate estimate of the amount of absorption exercised by the compound be judged, but the exact localities of such absorptions can be indisputably fixed.

* Bakerian Lecture "On the Photographic Method of Mapping the least Refrangible End of the Spectrum," by Captain W. DE W. ABNEY, R.E., F.R.S., Phil. Trans., 1880.

It appeared to us that the longer waves of the spectrum would be more likely to be affected by their passage through complex bodies than the shorter waves, and that therefore we ought to look for the absorption effects of the different molecular groupings in such bodies at the least refrangible end of the spectrum. The absorption spectra of various bodies in the ultra-violet portion of the spectrum have already been photographed by Professors HARTLEY and HUNTINGDON, and whilst that region seems specially adapted for certain series of hydro-carbon compounds, through a diminution or otherwise of the general absorption, &c., special or selective absorptions for all of them apparently lie at the opposite end of the spectrum. The results which we have tabulated in our maps fully support this view, indicating without much doubt that the substances we have examined can be grouped according to their absorption spectra; and that such grouping, as far as we have examined it, agrees on the whole with that adopted by chemists. It would be premature, however, to make any general deductions, since the ground to be covered is co-extensive with the compounds themselves; and as we have but made a selection which we believe may be regarded as typical, it will require the patient labour of many for a long period before this new branch of physico-chemical research can be in any way regarded as complete. We have thought that, by describing our method of work, and by publishing such results as we have already obtained, we might hasten the more extensive research which must eventually be undertaken. We have therefore the honour to bring the matter before the Society, with a statement that it is our intention as opportunity occurs to continue these investigations.

Apparatus.

Spectroscope.—The spectroscope employed was an excellently planned one by A. HILGER, consisting of three prisms of 62°, and a half prism to which was cemented a right-angled prism for the purpose of reflecting back the rays. The glass was white flint of medium density. The beam of light underwent total reflection at the half prism, so that the battery consisted in reality of one, three, five, or seven prisms, as occasion might require. The great advantage of this arrangement is that when once the camera is in position it does not require readjustment for any movement or alteration in the number of the prisms. There being, however, occasions when we rather doubted the performance of this arrangement, we adapted the spectroscope to the ordinary form without reflection. The slit of the spectroscope was furnished with an arrangement by which every third or half only of the aperture need be uncovered at a time. The collimating lens was of 20 inches focal length, and that of the lens fitted to the camera was of nearly the same, and in some cases of double that length. The dispersion of three prisms from A to D is about 3° 10′.

Camera.—The camera employed was made by Mr. P. MEAGHER from designs furnished by one of us, and is capable of being employed with any lens whose focal length lies between 9 inches and 6 feet. It has a horizontal swing back, which is

capable of swinging at a considerable angle to the axis of the lens—a point of importance when it is remembered what a rapid alteration there is in the focus of the rays, as they descend in the infra-red of the spectrum.

Reason for employing glass prisms.—From the nature of the research it was necessary to employ tubes to hold the various liquids, and it would have been impossible to have closed their ends with any material but glass owing to the solvent nature of some of them. From previous experience we knew that the glass used transmitted radiations to a wave-length of at least λ 20,000; and it would have been utterly impossible by prismatic analysis to have distinguished any except very general absorptions even as low down in the spectrum as that wave-length. For these reasons we determined to employ glass prisms. Our results show that in the majority of cases no advantage would have been gained by using rock salt; though for thin layers of vapour we can well imagine that rays of still lower refrangibility would have to be studied.

Number of prisms employed.—Three prisms were invariably employed when photographs were taken from which measurements had to be made. With one or two prisms the dispersion was insufficient to enable the details of the absorption spectra to be accurately observed; though the general character of the absorption was always clearly marked. With five prisms, on the other hand, the absorptions in some cases became too undefined; we therefore concluded that three prisms would be the best number with which to work. In nearly every case, however, photographs of the absorptions were taken with one or two prisms to give a preliminary idea of what we might expect to find with the greater dispersion. It is our intention to pursue the investigation with a diffraction grating, more especially to map the line spectra which have shown themselves.

Measurement of the photographs.—The measurement of the photographs was carefully made by means of a transparent scale applied to the film-side of the negatives, reading to $\frac{1}{4}$ a millimetre by division, and to half that quantity or less by estimation. We attempted to take the measures with a micrometer, but the nature of some of the absorptions precluded its use, since much magnification more or less shrouded the phenomena; with the solar spectrum where the absorptions are in more definite lines as opposed to bands, measurements with a high magnifying power are comparatively easy. The accuracy of our plan of measurement might be doubted, but it is really trustworthy after a little experience. We should not have trusted to it, however, without referring some of the principal spectra to a comparison with the solar spectrum. The wave-lengths of the FRAUNHOFER lines in the infra-red region were taken from the map furnished by one of us with the paper already referred to. A test of the accuracy of the measurements was the fact that the same wave-lengths of the absorption of any particular substance could always be obtained on several photographs.

We invariably compared all absorption spectra with that of ethyl iodide, which

furnishes almost entirely a line spectrum and, in fact, our measurements were referred to an empiric scale derived from the absorptions of this body, before we were fortunate enough to obtain a wave-length scale from strict comparison with the solar spectrum. We believe that the chloroform spectrum might be almost better to give a fiducial scale. The multiplicity of fine lines in both of these substances makes the mapping the spectra comparatively easy when once the position of those lines has been accurately fixed.

Source of light.—In the positive pole of the electric light we have a source of light which may be considered of uniform brightness, if the light emitted from the crater only be employed. Whether the crater be $\frac{1}{8}$ or $\frac{1}{2}$ inch in diameter (the one being produced by a small magneto-electric machine, and the latter by a larger one) the brightness remains constant, being apparently that due to the temperature at which carbon is vaporized. If then an image of the crater be projected on the slit it is evident that the spectrum produced will be a continuous one (crossed, perhaps, in the more refrangible regions by bright lines) and of constant brightness. It is worthy of remark that in no case have we found the continuous spectrum in the least refrangible region to have any signs of bright or dark lines when the crater is projected on the slit. We discuss further on the possibility of any of the lines mapped being due to the carbon vapour in the arc. In order to get as large a portion as possible of the crater available for projection on the slit, the lower pole (which in this case was the positive) was placed slightly behind the upper pole. The electric light we employed was generated by an "M." Gramme machine, driven by a small three horse-power Brotherhood engine, of whose performance we cannot speak too highly. The current is of such a tension that it is capable of supporting an arc of some $\frac{1}{4}$-inch in length. The lamp employed was a hand-lamp of the Sautter Lemonier's pattern, and for general photographic spectrum work is everything that can be desired. For photographic purposes the radiation emitted from a gas flame is much too feeble to be of any service, and there is no economy in using the oxy-hydrogen light when the electric light is available.

Tubes for holding the liquids.—Tubes of 2 feet, 6 inches, and 3 inches respectively were used for the more common and inexpensive liquids, but for those which were more difficult to obtain and costly the two shorter tubes were invariably used. The longer tubes had diameters of 2 or $1\frac{1}{2}$ inches, and the shorter of $\frac{3}{4}$-inch and $\frac{1}{2}$-inch. In order to get a bright image on the slit, and at the same time to cover the whole aperture, the carbon points were placed about 3 feet from the collimator. The diameter of the condensing lens was about $2\frac{1}{2}$ inches, which thus just cast a beam sufficient to fill the collimating lens. The smaller tubes were placed in the position where the rays crossed, and thus when they were interposed before the slit the brightness suffered no diminution except that due to the liquid itself.

Photographic arrangements.

Exposures.—The exposures were arranged in accordance with what experience taught us was the best time for different lengths of tube and differing liquids. In every case photographs were taken of each compound with varying lengths of exposure, in order to ascertain which gave the most trustworthy and measurable results. On the one hand, it was necessary to show the detail in those parts which had no general absorption but in which there was special absorptions; and, on the other hand, it was necessary to show the special absorptions in those regions which had general absorption superposed. These details were often obtained by means of one or more photographs. It had also to be remembered that the silver salt employed, though sensitive, it is believed, to the extreme theoretical limit of the prismatic spectrum, yet has one maximum of sensitiveness, situated a little below the limit of the visible spectrum, from which point the "curve of sensitiveness" gradually falls away on each side. Thus, with an unabsorbed continuous spectrum it would require about three times more exposure for λ 12,000 to impress the film to develop to a standard density than it would for λ 8000. This difficulty is, however, met with at the other end of the spectrum, where the usual photographic plates are employed, and is necessarily present in all spectrum photography.

Precautions to be observed.—There are a few cautions to give to other observers—first, to ascertain that the floor of the laboratory is free from vibration; second, to get an accurate focus of the locality near λ 900; third, to set the prisms at an angle of minimum deviation for that locality, since in that region the greatest variations in the absorption phenomena are to be found; fourth, to use a good reference spectrum, such as that of chloroform; and, fifth, to see that the tubes are accurately centered with the axis of the collimating tube.

Mapping the results.

In order to map the results in wave-lengths, photographs of the solar spectrum were made with the same spectroscope as that employed in the research, and the FRAUNHOFER lines accurately measured and laid down to scale. The wave-lengths, also to scale, were erected as ordinates and a curve constructed. The value for every 10 in the wave-length scale was then found on the curve and fresh ordinates plotted from these points on the curve, and this was our "prismatic scale." When plotting an absorption spectrum the distance between two known points was noted off on a convenient ivory scale, and this applied to the ordinates of the prismatic scale at the known wave-lengths. All intermediate points could then be immediately read off in wave-lengths. The wave-lengths between 900 and 1000 we consider to be accurate to within one or two, and beyond that point to within three or four; the shorter wave-lengths are accurate within one, we believe.

Substances examined.

We append a list of substances the absorption spectra of which we have mapped on an empiric scale reduced to wave-lengths subsequent to the reading of the paper, but it must not be inferred that these are all that we have examined. There are various aqueous solutions which we have not dealt with in this paper, preferring to eliminate as far as possible any matter which did not seem to bear directly on the objects we had in view. We have, in addition to the map, made a tabular statement of each spectrum, inserting approximate wave-lengths, which we trust will be of use to future observers.

Different kinds of absorption.

It may be useful to state the different kinds of absorption with which we have met.

1st. General absorption from the least refrangible end.

2nd. Special absorptions
- Lines
  - fuzzy, or
  - sharp.
- Bands
  - both edges sharply defined.
  - one edge sharply defined.
  - both edges not sharply defined.

Regarding the general absorption we have nothing very noteworthy to remark, beyond the fact that as a rule in the hydrocarbons of the same series those of heavier molecular constitution seem to have less than those of lighter.

In regard to the special absorptions we may say that it is very often easy to distinguish the position of lines at the edges of bands, though they are not sufficiently marked to be represented on the map. This is more noteworthy, when we have a change from a very dense absorption to a feebler one. The impression left on the mind is that in reality these particular bands are made up of lines of different intensities, but of this we cannot speak with authority.

As to the possibility of any of these special absorptions being due to the carbon vapour, &c., of the arc, it will, we think, be sufficient to state that, as we do not find any trace of them in photographs taken without a liquid in front of the slit, we are quite satisfied that they can be due only to the substances with which we have experimented.

Nature of the absorptions.

As regards the nature of the absorptions caused by the different substances experimented on, we started with no preconceived idea—we simply mapped what we measured. In our earlier photographs, which were of substances containing oxygen, we found the absorptions to be mostly in bands and irregularly-placed and ill-defined lines, the want of definition we now know being caused by maladjustment of the focus and a want of rigidity in our laboratory floor. It was not until we had removed our

apparatus to a more stable site, and till we examined some of the iodides of the alcohol series, that we observed many sharply-defined lines and also that the edges of the bands were more rigidly marked; and when it appeared that some of the edges of the bands in the compounds containing oxygen were coincident with some of the fine lines in the iodides, we were forced to the conclusion that there must be some connexion between the one and the other, since such an agreement could not be fortuitous. Our endeavour was therefore to discover, if possible, what constituents of the iodides caused these lines. They must be due to carbon, iodine, or hydrogen, or to a combination of one with the other.

Our first step was to get a substance which should contain but one atom of carbon and one of hydrogen, and this we found in chloroform. On examining the photograph of this spectrum we were gratified to find that all bands had disappeared, and that the absorption spectra contained only lines, some fine and some broad. By reducing, then, the carbon and hydrogen and increasing the halogen a line spectrum was produced.

It seemed possible, nay, probable, that these lines might be due to the chlorine present in such abundance in the chloroform, and if so it was evident that by abstracting the last atom of hydrogen and taking carbon tetrachloride an absorption spectrum of a still simpler form, but still lineal, should be obtained. To our surprise, however, we found that the absorption of carbon tetrachloride was an absolute blank, and that there was no absorption beyond a slight general one at the least refrangible end of the infra-red. With carbon disulphide the same negative result was also obtained.

This seemed to show that neither chlorine nor carbon had anything to do with the linear spectrum observed in chloroform. Dr. HODGKINSON kindly prepared for us a solution of cyanogen in carbon tetrachloride (180 vols. of gas in one of the solvent), and not a trace of line or band was found in its spectrum. A crucial test was to observe spectra containing hydrogen and chlorine, hydrogen and oxygen, and hydrogen and nitrogen.

We therefore tried hydrochloric acid and obtained a spectrum containing some few lines. Water gave lines, together with bands, two lines being coincident with those in the spectrum of hydrochloric acid.

In ammonia, nitric acid, and sulphuric acid we also obtained sharply-marked lines, coincidences in the different spectra being observed, and nearly every line mapped found its analogue in the chloroform spectrum, and usually in that of ethyl iodide. Benzine again gave a spectrum consisting principally of lines, and these were coincident with some lines also to be found in chloroform. It seems then that the hydrogen, which is common to all these different compounds, must be the cause of the linear spectrum. In what manner the hydrogen annihilates the waves of radiation at these particular points is a question which is at present, at all events, an open one, but that the linear absorptions, common to the hydrocarbons and to those bodies in which hydrogen is in combination with other elements such as oxygen and nitrogen, is due to hydrogen there can be no manner of doubt.

We may go a step further than this, however. We find both edges of some bands to accord with the position of known hydrogen lines, whilst in others we find that only one edge can be so marked. Though direct evidence is wanting to enable us to say that the other edge marks the position of a hydrogen line, yet the circumstantial evidence that such is the case is excessively strong. In point of fact, of the hydrogen lines and edges of bands to be found in hydrocarbons lying between 900 and 972 of our empiric scale, more than half are to be found coincident with lines in the non-carbon bodies. The following table shows the coincidences :—

Hydrocarbons	900	902	905	910	912	920	927	935	942	949	952	959	972
Hydrogen and oxygen, or nitrogen, or chlorine	900	..	..	..	912	920	927	..	..	949	..	959	972

On the more refrangible side of 900 the coincidences in the latter series are always to be found in the former. If other bodies containing no carbon be examined no doubt some of the gaps in the table will be filled up. It must distinctly be understood that in all the absorptions in which bands, lines, or both appear, the position of the whole of the known hydrogen lines will not be found, each weighted radical making a selection of them.

Effect of the presence of oxygen.

It is seen that bodies containing carbon and hydrogen alone or with chlorine, bromine, or iodine, gave absorption spectra in which there are defined bands together with lines.

The next point that required solution was the effect of the presence of oxygen on the body under examination, and here we had ample material on which to make our observations. It appears that in every case where oxygen is present otherwise than as a part of the radical it is attached to some hydrogen atom in such a way that it obliterates the radiation between two of the lines which are due to that hydrogen. Take, for example, ethyl alcohol. We find that one band of absorption takes place between 927 and 942, another between 900, and 905 on the less refrangible, and 892 on the more refrangible side. Now, all these different numbers are localities where hydrogen lines are to be found. Iso-butyl alcohol is another good example. Besides the last-named bands of the ethyl alcohol it has bands lying between 912 and 920, between 927 and 942, and a narrow band about 959. These, again, are all localities where hydrogen lines can exist. If more than one hydroxyl group be present we doubt if any different effect is produced beyond that produced by one hydroxyl group, except a possible greater general absorption ; a good example of this will be found in cinnamic alcohol and phenyl-propyl alcohol, which give the same spectra as far as the special absorptions are concerned.

Effect of increase or diminution in the length of the absorbing medium.

A natural question to ask is as to the effect of the increase or the diminution in length of the compound placed before the slit of the spectroscope. The answer is not difficult to give. Where an increase of length is used, one of two things occurs: either general absorption creeps up further towards the more refrangible end, or the absorption features are more marked. It may be supposed that in the latter case the bands should become more defined, but this is not so; the bands, as the length of column of liquid increases, may spread out till they reach the locale of another hydrogen band, each line becoming, as it were, a stepping stone for a further advance of absorption. This usually takes place only on the more refrangible edge of a band, the less refrangible edge remaining, as a rule, constant. In some cases both edges of the band remain fixed (as example we may quote the ethyl alcohol band situated about 900) and neither increase nor diminution of length of fluid alters their relative positions.

Where the length of column is diminished one of three things happens: the absorption disappears altogether, the bands fade into the lines bounding one of their edges, or they become fainter and remain constant. At the edge of the band which is least defined the absorption gradually disappears till a line is left at the most defined edge, or if both are ill-defined the nucleus will probably be found to be central to it. When both edges are well-defined the band remains constant in width, but fainter.

Oxygen combined in the radical.

Hitherto we have only taken into account oxygen which is not contained in the radical; when it is so contained it appears to act differently, always supposing hydrogen to be present as well. We need only refer to the spectrum of aldehyde which is inclined to be linear rather than banded, or rather the bands are bounded by absolute lines, and are more defined than when oxygen is more loosely bonded.

Detection of the radical.

An inspection of our maps will show that the radical of a body is represented by certain well-marked bands, some differing in position according as it is bonded with hydrogen, or a halogen, or with carbon, oxygen, or nitrogen. There seem to be characteristic bands, however, of any one series of radicals between 1000 and about 1100, which would indicate what may be called the central hydrocarbon group, to which other radicals may be bonded.

The clue to the composition of a body, however, would seem to lie between λ 700 and λ 1000. Certain radicals have a distinctive absorption about λ 700 together with others about λ 900, and if the first be visible it almost follows that the distinctive mark of the radical with which it is connected will be found. Thus in the ethyl series we find

an absorption at 740, and a characteristic band one edge of which is at 892, and the other at 920. If we find a body containing the 740 absorption and a band with the most refrangible edge commencing at 892, or with the least refrangible edge terminating at 920, we may be pretty sure that we have an ethyl radical present. So with any of the aromatic group; the crucial line is at 867. If that line be connected with a band we may feel certain that some derivative of benzine is present. The benzyl group show this remarkably well, since we see that phenyl is present, as is also methyl. It will be advantageous if the spectra of ammonia, benzine, aniline, and dimethyl aniline be compared, when the remarkable coincidences will at once become apparent, as also the different weighting of the molecule. The spectrum of nitro-benzine is also worth comparing with benzine and nitric acid. We should have liked to have said more regarding the detection of the different radicals, but it might seem presumptuous on our part to lay down any general law on the results of the comparatively few compounds which we have examined. In our own minds there lingers no doubt as to the easy detection of any radical which we have examined, but it will require more energy and ability than we possess to thoroughly classify all the different modifications which may arise.

We may say, however, it seems highly probable by this delicate mode of analysis that the hypothetical position of any hydrogen which is replaced may be identified, a point which is of prime importance in organic chemistry.

The detection of the presence of chlorine or bromine or iodine in a compound is at present undecided, and it may well be that we may have to look for its effects in a different part of the spectrum. The only trace we can find at present is in ethyl bromide, in which the radical band, about 900, is curtailed in one wing. The difference between amyl iodide and amyl bromide is not sufficiently marked to be of any value. We quit this part of our subject in the hope that chemists will be able to help us to decipher more than has as yet been done.

Solar coincidences.

We have already stated that in order to determine the wave-lengths of the absorptions in the different spectra, we have taken photographs of the solar spectrum on the same plate with that of a few of the principal substances. Unfortunately, when at first we took up the iodides we had no opportunity of thus obtaining a direct comparison between the FRAUNHOFER lines and ethyl iodide; but as a matter immediately affecting our results this has nothing to do, since all the substances mapped are mapped in reference to ethyl iodide. When our maps were at the point of completion, and in fact when all the measurements but two or three had been made, it struck us that we might find some analogy between the solar spectrum of this region and our linear absorptions, since these are presumably due to hydrogen. At that time we noted several of what appeared to be coincidences

between the absorptions in the organic compounds and in the solar spectrum. Since the reading of the paper* we have further investigated these coincidences with greater prismatic dispersion and also with a diffraction grating of 17,260 lines to the inch. The indubitable result is that the broad line in the solar spectrum at λ 866 is coincident with the radical line of benzine which in our scale is 856. Comparing the ethyl iodide group with the solar spectrum we find that the following bands are coincident with bands or lines in the solar spectrum: 880 to 888 with λ 895 to λ 903, 888 to 892 with λ 904 to 907, 898 to 902 with λ 913 to 918, 902 to 912 with λ 927 to 930·5, 912 to 920 with λ 932 to 942. These bands are essentially the radical bands of the ethyl series. There are some more apparent solar coincidences in the spectra of water and chloroform. It is very remarkable that the line 856 in our scale should be the basic lines of the benzine series. When the thickness of benzine is gradually diminished this is the last line which disappears, but it remains of constant breadth to the end. Should it appear by subsequent investigations that the ethyl radical is really to be found in the solar spectrum it might be due to acetylene, from which immediately would follow the formation of benzine.

Again, anyone who looks at a photograph of the solar A band and that of benzine, must be struck with the close structural likeness between the two; it would not be at all surprising to find that λ 760 was another nucleus for a hydrocarbon group.

The X group of lines in the solar spectrum and that of the groups already mentioned appears to one of us to remain of constant intensity at any elevation of the sun or in any state of the intermediate atmosphere. It may therefore be necessary to refer them to some absorbing medium lying at the sun itself, or beyond the usually accepted limits of our atmosphere. Be it where it may, the fact remains that, in two instances at least, a study of the absorption spectra of organic bodies has to some extent thrown a glimmering of meaning on some of the absorption lines of the solar spectrum.

The value of a study of the absorption spectra of liquids (not including in this term solids in solution) in order to gain an insight into their molecular constitution has been demonstrated by Professors HARTLEY and HUNTINGDON, and by their organized attack they have thrown much light on the subject; but we venture to think that the results we have obtained will prove that in these absorptions a still greater insight into the molecular constitutions of such bodies may be given. We may be too sanguine, but we believe not. It seems to us that the spectra leave as definite characters to read as are to be found in hieroglyphics, and we venture to think that we have given a clue to enable them to be deciphered. As to our theory that the foundation of all absorptions in these bodies is the hydrogen, we are content to leave it for discussion. If it should prove unreliable, the *clue* alone to the characters must be sought elsewhere: the story is still to be read by any one who may find a better one and a truer. Fully conscious of this, we would invite an exhaustive discussion, deeming that the results alone, which are indisputable, will give a ready basis for it.

* March, 1881.

In conclusion, we have to say that we are indebted to Dr. FRANKLAND, Dr. ARMSTRONG, Dr. RUSSELL, and Dr. GUTHRIE for several organic substances; but to Dr. HODGKINSON are our thanks especially due for the ready manner in which he lent us many choice specimens of his own preparation, and for the kind advice he was at all times ready to offer us (and which we freely accepted) as to the kind of compound to examine which would most aid us at any particular stage of our research. It is almost superfluous to mention that our work has been largely aided in the preparation of photographic material by Sergeant JACKSON, R.E., and we take this opportunity of thanking him for the least interesting portion of the work which he so willingly carried out for us. The labour may be appreciated when it is considered that many hundreds of photographs have been taken, some 300 of which have been measured and compared. A research of this kind is always arduous, but it has been lightened by the cordial cooperation of those whom we have named. Our thanks are also due to Mr. DICK for the careful way in which he has drawn our maps.

APPENDIX.

The following appendix contains a tabular statement* of all the lines and bands measured in the different compounds mapped; where not otherwise expressly stated, 6 inches was the length of the column of liquid used.

The intensities of the lines have been taken as 1, $\frac{3}{4}$, $\frac{1}{2}$, and $\frac{1}{4}$: the first-named intensity signifying total transparency in the photographs. When an intensity is shown as $\left.\begin{matrix}0 \text{ to } 1\\1 \text{ to } 0\end{matrix}\right\}$ it means that there is a gradually increasing intensity of absorption between the first-named edge of a band and the centre of it, and that it decreases at an equal rate on the other side of the centre to the second-named edge of the band. When a band is shown as of one intensity (say $\frac{1}{2}$), it means that the whole of that band has an equal intensity throughout. When it is shown as (say) 0 to 1 it means that the intensity increases from 0 at the first edge of the band and terminates abruptly at the second edge. These explanations, it is believed, will suffice to make the appendix clear. The sources from whence the substances analysed were obtained are shown in the tables.

* The approximate wave-lengths have been added since the paper was read, March, 1881.

Compound.	Lines (L) and bands (B).			
	Approximate wave-length.	Scale on plates.	Intensity.	
Methyl alcohol (Kahlbaum).	867	857 L	$\frac{1}{2}$	
	905 }	890 } B	$\frac{1}{4}$	
	913 }	898 }		
	913 }	898 }		
	917 }	902 } B	1	
	917 }	902 }		
	926 }	908 } B	$\frac{3}{4}$	
	926 }	908 }		
	930 }	912 } B	1	
	930 }	912 }		
	942 }	920 } B	$\frac{3}{4}$	
	942 }	920 }		
	949 }	927 } B	1	
	972	947 L	1	
	985	959	1	Broad line.
	985 }	959 } B	$\frac{1}{2}$	
	1017 }	985 }		
	1017 }	985 }		
	1020 }	987 }	$\frac{1}{2}$ to 1	
Ethyl alcohol (Kahlbaum).	741	740 L	$\frac{1}{2}$	Fuzzy line.
	885	872 L	$\frac{1}{2}$	
	907 }	892 } B	{ 0 to 1 }	1 is reached about 902.
	922 }	905 }	{ 1 to 0 }	
	949	927 L	1	
	949 }	927 } B	0 to 1	
	967 }	943 }		
	985	959 L	1	Broad.
	986 }	960 } B	$\frac{1}{2}$	
	1153 }	1090 }		
	1066	1025 B	1	As in ethyl iodide.
	1086	1037 B	1	" "
	1135	1072 B	1	" "
	1165	1100	1	End of spectrum.
Propyl alcohol (Kahlbaum).	746	745 L	$\frac{1}{2}$	Broad line.
	895	880 L	$\frac{1}{2}$	
	908 }	892 } B	0 to $\frac{1}{4}$	
	913 }	898 }		
	913 }	898 }		
	917 }	902 } B	1	
	917 }	902 }		
	942 }	920 } B	1 to 0	
	985	959 L	B	
	985 }	959 } B	$\frac{1}{2}$	
	— }	1160 }		
	1040	1003 B	1	As in ethyl iodide.
	1066	1025 B	1	" "
	1086	1037 B	1	" "
	1135	1072 B	1	" "
	1165	1100 B	1	" "

Compound.	Lines (L) and bands (B). Approximate wave-length.	Scale on plates.		Intensity.	
Isopropyl alcohol (Kahlbaum).	741	740	L	½	Fuzzy, extending from 734 to 742.
	885	872	L	½	
	907 }	892 }	B	{ 0 to 1 }	As in ethyl alcohol.
	922 }	905 }		{ 1 to 0 }	
	949 }	927 }	B	0 to 1	
	966 }	942 }			
	985	959	L	1	Broad line.
	1040	1003	L	1	" "
	1040 }	1003 }	B	0 to 1	
	1061 }	1020 }			
	1061	1020		1	End of spectrum.
Pseudo butyl (Kahlbaum).	735	735	L	½	
	741 }	740 }	B	½	
	746 }	745 }			
	895	880	L		
	907 }	892 }	B	0 to ½	
	913 }	898 }			
	913 }	898 }	B	1	
	922 }	905 }			
	930 }	912 }	B	1	
	942 }	920 }			
	949 }	927 }	B	1	
	966 }	942 }			
	985	959	L	1	Broad.
	1020 }	987 }	B	0 to ½	
	1066 }	1025 }			
	1066 }	1025 }		½	
	— }	1200 }			
	1066	1025	B	1	As in ethyl iodide.
	1087	1038	B	1	" "
	1165	1100	B	1	" "
	—	1200	L	1	End of spectrum.
Isobutyl (Kahlbaum).	741 }	740 }	B	½	
	748 }	747 }			
	895	880	L		
	907 }	892 }	B	0 to ½	
	904 }	889 }			
	913 }	898 }	B	1	
	922 }	905 }			
	930 }	912 }	B	1	
	942 }	920 }			
	949 }	927 }	B	1	
	966 }	942 }			
	985	959	L	1	Broad.
	1020	987	L	1	
	1020 }	987 }	B	0 to 1	
	1036 }	1000 }			
	1036	1000		1	End of spectrum. [With prolonged exposure traces of bands at 925, 937, 972, and 1100 are visible.]

Compound.	Lines (L) and bands (B).			
	Approximate wave-length.	Scale on plates.	Intensity.	
Amyl alcohol (Frankland).	753	752 L	$\frac{1}{2}$	
	893	878 L	$\frac{1}{2}$	
	913 }	898 } B	$\frac{1}{2}$	
	923 }	905 }		
	923 }	905 } B	1	
	927 }	910 }		
	927 }	910 } B	$\frac{1}{2}$	
	942 }	920 }		
	949 }	927 } B	0 to $\frac{3}{4}$	
	973 }	948 }	$\frac{3}{4}$ to 0	
	973 }	948 } B	$\frac{1}{4}$	
	985 }	959 }		
	985 }	959 } B	0 to $\frac{3}{4}$	
	994 }	967 }		
	994 }	967 } B	$\frac{3}{4}$	
	1002 }	972 }		
	1002	972 L	1	Broad line.
	1020	987 L	1	,, ,,
	1066	1025 B	1	As in ethyl iodide.
	1087	1038 B	1	,, ,,
	1135	1072 B	1	,, ,,
	1165	1100 B	1	
	1066 }	1025 } B	$\frac{1}{2}$	Background.
	— }	1160 }		
	—	1160 }		
	—	1180 }	$\frac{1}{2}$ to 1	,, ,,
	—	1180	1	End of spectrum.
Methyl iodide (Kahlbaum).	720	722 L	1	Broad with nucleus.
	867	857 L	1	This line forms a nucleus between a band of $\frac{1}{2}$ intensity extending from 852 to 862.
	885 }	872 } B	0 to 1	
	892 }	878 }		
	892 }	878 }		
	899 }	884 } B	1	
	915	900 L	1	Very fine.
	1020	987 L	1	Broad.
	1040	1003 L	1	Broader.
	1066	1025 B	1	As in ethyl iodide.
	1086	1037 B	1	,, ,,
	1135	1072 B	1	,, ,,
	1165	1100 B	1	,, ,,
	1066 }	1025 } B	$\frac{1}{2}$	
	— }	1120 }		
	—	1120 }		
	—	1140 } B	$\frac{1}{2}$ to 1	
	—	1140	1	End of spectrum.

Compound.	Lines (L) and bands (B). Approximate wave-length.	Scale on plates.	Intensity.	
Ethyl iodide, standard (FRANKLAND).	732	732 L	1	Broad with nucleus.
	741	740 L	½	
	774	772 L	½	
	782	778 L	½	
	813	808 L	¼	
	833	824 L	¼	
	867	857 L	1	N.B.—From 1025 to 1180 the background is ½ and from 1180 to 1200 0 to 1, near which point is end of spectrum.
	895 }	880 } B	0 to 1	
	903 }	888 }		
	903 }	888 } B	1	
	907 }	892 }		
	913 }	898 } B	1	
	918 }	902 }		
	928 }	908 } B	1	
	930 }	912 }		
	930 }	912 } B	1 to 0	
	942 }	920 }		
	949	927 L	½	
	958	935 L	½	
	966	942 L	½	
	972	947 L	½	
	977	952 L	½	
	985	959 L	1	
	1002	972 L	1	Broad.
	1020	987 L	1	"
	1040	1003 L	1	Broader.
	1061 }	1020 } B	½ to 1	
	1075 }	1030 }	1 to ½	
	1082 }	1035 } B	½ to 1	
	1093 }	1042 }	1 to ½	
	1124 }	1065 } B	½ to 1	
	1143 }	1078 }	1 to ½	
	1160 }	1095 } B	1 to 1	
	1171 }	1105 }		
Ethyl bromide (KAHLBAUM).	..	..	..	This spectrum is the same as ethyl iodide, except that the bands 908 to 912, and 912 to 920 are replaced by—
	926 }	908 } B	1	
	933 }	914 }		
	933 }	914 }	½	
	942 }	920 }		

Compound.	Lines (L) and bands (B).			
	Approximate wave-length.	Scale on plates.	Intensity.	
Propyl iodide (KAHLBAUM).	732	732 L	½	
	746	745 L	½	
	867	857 L	½	
	902	886 B	0 to 1	
	910	895 B	1 to 0	
	917	902 B	0 to 1	
	925	907 B	1 to 0	
	947	927 B	0 to 1	
	966	942 B	1 to 0	
	985	959 L	½	
	1002	972 L	½	
	1020	987 L	1	As in ethyl iodide.
	1040	1003 L	1	" "
	1066	1025 B	½	
	—	1160 B		
	1066	1025 B	1	" "
	1087	1038 B	1	" "
	1135	1072 B	1	" "
	1165	1100 B	1	" "
Amyl iodide or amyl bromide (KAHLBAUM).	732	732 L	½	Shade ½ between these lines.
	752	751 L	½	
	867	857 L	½	
	886	872 L	½	
	895	880 B	½	
	907	892 B		
	917	902 B	1	
	930	912 B		
	930	912 B	½	
	942	920 B		
	957	935 B	0 to 1	
	971	947 B	1 to 0	
	1020	987 L	½	
	1040	1003 L	½	Broader.
	1066	1025 B	1	As in ethyl iodide.
	1086	1037 B	1	" "
	1135	1072 B	1	" "
	1165	1100 B	1	" "
	1066	1025 B	½	
	—	1200 B		
Hexyl iodide (KAHLBAUM).	748	747 L	½	
	867	857 L	½	
	886	872 L	½	
	900	884 B	0 to ¾	
	907	892 B		
	907	892 L	1	Shade between these lines.
	915	900 L	1	
	915	900 B	¾ to 0	Gradual shade.
	966	942 B		
	947	927 L	¾	
	1020	987 L	½	
	1020	987 B	0 to ½	
	1061	1020 B		
	1040	1003 L	¾	
	1066	1025 B	1	As in ethyl iodide.
	1087	1038 B	1	" "
	1135	1072 B	1	" "
	1165	1100 B	1	" "

Compound.	Lines (L) and bands (B). Approximate wave-length.	Scale on plates.	Intensity.	
Aldehyde (Kahlbaum).	732	732 L	½	Rather a fuzzy line.
	867	857 L	½	
	895 } B	880 } B	0 rapid to ½ and thence gradual to ¾ at 892	
	907 }	892 }		
	911 } B	895 } B	Steep from 0 to ¾, gradual to ½, and steep to 0	
	925 }	908 }		
	947	927 L	½	
	971 } B	947 } B	Band shaded equally 0 to ½ and ½ to 0, 959 L	
	1020 }	987 }		
	985	959 L	¾	
	1020	987 L	¾	
	1040 } B	1003 } B	Shaded equally from 0 to ½ and ½ to 0	
	1090 }	1040 }		
	1047 } B	1010 } B	1	This band appears through the wider previous band.
	1061 }	1020 }		
	1135	1072 B	1	A band as in ethyl iodide.
	1165	1100 B	1	N.B.—From 1040 to 1120 background ½, and end of spectrum at 1160.
Paraldehyde (Kahlbaum).	732	732 L	½	
	867	857 L	½	
	895 } B	880 } B	0 to 1 at 892	
	913 }	898 }	1 to 0 at 898	
	980 to 1177 } B	955 to 1110 } B	Gradual shade of background 0 to ¾	
	980 } B	955 } B	0 to ½	
	992 }	963 }	½ to 0	
	983	959 L	1	
	998 } B	970 } B	0 to ½	
	1004 }	974 }	½ to 0	
	1001	972 L	1	
	1040	1003 L	1	
	1040 } B	1003 } B	0 to 1	
	1061 }	1020 }	1 to 0	
	1061	1020 L	1	
	1090 } B	1040 } B	0 to 1	
	1103 }	1050 }	1 to 0	
	1143 } B	1080 } B	0 to 1	
	1156 }	1090 }	1 to 0	
	1162 } B	1095 } B	0 to 1	
	1171 }	1105 }	1 to 0	
	1177	1110	1	End of spectrum.

Compound.	Lines (L) and bands (B).			
	Approximate wave-length.	Scale on plates.	Intensity.	
Diethyl ether (Hopkin and Williams)	741	740 L		
	907 }	892 } B	1	
	920 }	903 }		
	958 }	935 } B	0 to 1	
	985 }	959 }		
	1040 }	1003 } B	{ 0 to 1 }	1 at 1025.
	1075 }	1030 }	{ 1 to 0 }	
	1086	1037 B	1	As in ethyl iodide.
	1135	1072 B	1	" "
	1165	1100 B	1	" "
	1075 }	1030 } B	$\frac{1}{2}$	Background.
	— }	1130 }		
	—	1130 } B	$\frac{1}{2}$ to 1	
	—	1140 }		
	—	1140	1	End of spectrum.
Amyl ether (Kahlbaum).	753	751 L	$\frac{1}{2}$	
	893	878 L	$\frac{1}{2}$	
	913 }	898 } B	$\frac{1}{2}$	
	922 }	905 }		
	922 }	905 } B	1	
	927 }	910 }		
	927 }	910 } B	$\frac{1}{2}$	
	942 }	920 }		
	942 }	920 } B	$\frac{1}{2}$ to 0	
	949 }	927 }		
	966 }	942 } B	{ 0 to 1	
	1002 }	972 }	{ 1 to 0	
	1040	1003 L	1	Broad.
	1066	1025 L	1	As in ethyl iodide.
	1040 }	1003 } B	$\frac{1}{4}$	
	1066 }	1025 }		
	1066 }	1025 } B	$\frac{1}{2}$	
	— }	1175 }		
	1139 }	1075 } B	{ 0 to 1	
	1150 }	1085 }	{ 1 to 0	
	1165	1100 B	1	As in ethyl iodide.
	—	1175 } B	$\frac{1}{2}$ to 1	
	—	1180 }		
	—	1180	1	End of spectrum.

Compound.	Lines (L) and bands (B).			
	Approximate wave-length.	Scale on plates.	Intensity.	
Ethyl nitrate (Kahlbaum).	741	740 L	$\frac{1}{2}$	
	873	862 L	$\frac{1}{2}$	
	896 }	882 } B	$\frac{1}{4}$	
	907 }	892 }		
	907 }	892 } B	1	
	915 }	900 }		
	952 }	930 } B	0 to $\frac{3}{4}$	
	963 }	940 }	$\frac{3}{4}$ to 0	
	996 }	968 } B	0 to $\frac{3}{4}$	
	1006 }	976 }	$\frac{3}{4}$ to 0	
	1020	987 L	$\frac{1}{2}$	
	1036 }	1000 } B	0 to $\frac{3}{4}$	
	1051 }	1013 }	$\frac{3}{4}$ to 0	
	1096 }	1045 } B	0 to $\frac{3}{4}$	
	1108 }	1053 }	$\frac{3}{4}$ to 0	
	1159 }	1095 } B	0 to $\frac{3}{4}$	
	1171 }	1105 }	$\frac{3}{4}$ to 0	
	1040 }	1003 }		
	— }	to } B	$\frac{1}{2}$	
	—	1160 }		
	—	1160 } B	$\frac{1}{2}$ to 1	
	—	1165 }		
	—	1165	1	End of spectrum.
Ethyl oxalate (Hodgkinson.)	741	740 L	$\frac{1}{2}$	
	873	862 L	$\frac{1}{2}$	
	896 }	882 } B	0 to $\frac{1}{2}$	
	907 }	892 }		
	907 }	892 } B	1	
	915 }	900 }		
	952 }	930 } B	0 to $\frac{3}{4}$	
	963 }	940 }	$\frac{3}{4}$ to 0	
	1036 }	1000 } B	0 to $\frac{3}{4}$	
	1051 }	1013 }	$\frac{3}{4}$ to 0	
	1096 }	1045 } B	0 to $\frac{3}{4}$	
	1110 }	1055 }	$\frac{3}{4}$ to 0	
	1159 }	1095 } B	0 to $\frac{3}{4}$	
	1171 }	1105 }	$\frac{3}{4}$ to 0	
	952 }	930 } B	$\frac{1}{2}$	
	— }	1160 }		
	—	1160 } B	$\frac{1}{2}$ to 1	
	—	1165 }		
	—	1165	1	End of spectrum.

Compound.	Lines (L) and bands (B).			
	Approximate wave-length.	Scale on plates.	Intensity.	
Ethyl sulphide (Frankland).	741	740 L	1	
	867	857 L	1	
	886	872 L	$\frac{1}{2}$	
	907	892 B	0 to $\frac{3}{4}$	
	913	898 B		
	913	898 B	1	
	917	902 B	$\frac{3}{4}$ to $\frac{1}{2}$	
	917	902 B		
	922	905 B		
	922	905 B	$\frac{1}{2}$ to 0	The shade from $\frac{1}{2}$ commences about 919.
	942	920 B		
	1030	995 B	0 to $\frac{1}{2}$	
	1075	1030 B		
	1040	1003 L	1	
	1061	1020 B	$\frac{1}{2}$	
	—	1160 B		
	1067	1025 B	1	As in ethyl iodide.
	1087	1038 B	1	" "
	1135	1072 B	1	" "
	1165	1100 B	1	" "
	—	1160	1	End of spectrum.
Amyl nitrite (Guthrie).	753	752 L	$\frac{1}{2}$	
	886	872 L	$\frac{1}{2}$	
	907	892 B	$\frac{1}{2}$	
	915	900 B		
	915	900 B	1	
	928	910 B		
	928	910 B	$\frac{1}{2}$	
	942	920 B		
	966	942 L	$\frac{1}{2}$	Broad.
	985	959 L	1	"
	1002	972 L	$\frac{1}{2}$	
	1020	987 L		
	1040	1003 B	1	As in ethyl iodide.
	1067	1025 B	1	" "
	1086	1037 B	1	" "
	1135	1072 B	1	" "
	1165	1100 B	1	" "
	1040	1003 B	$\frac{1}{2}$ to 1	
	—	1120 B		
Formic acid (Frankland).	732	732 L	$\frac{1}{2}$	Broad line of which 732 is the distinct nucleus.
	893	878 B	$\frac{1}{2}$	
	930	912 B		
	930	912 B	$\frac{1}{4}$	
	966	942 B		
	966	942 B	$\frac{1}{4}$ to 1	
	973	948 B		
	968	945		End of spectrum.

Compound.	Lines (L) and bands (B).			
	Approximate wave-length.	Scale on plates.	Intensity.	
Glacial acetic acid (Kahlbaum).	887 } 900 }	875 } B	{ 0 to 1 { 1 to $\frac{1}{4}$	
	900 } 907 }	884 } B	$\frac{1}{4}$	
	907 } 913 }	892 } B	{ 0 to 1 { 1 to 0	
	930	912 L	$\frac{1}{2}$	
	930 } 949 }	912 } B	{ 0 to $\frac{1}{2}$ { $\frac{1}{2}$ to 0	
	949	927 L	$\frac{1}{2}$	
	966 } 985 }	942 } B	{ 0 to 1 { 1 to $\frac{1}{2}$	
	985 } 1010 }	959 } B	$\frac{1}{2}$	
	1010 } 1024 }	980 } B	$\frac{1}{2}$ to 1	
	1024	990	..	End of spectrum.
Propionic acid. (Kahlbaum).	736	736 L	$\frac{1}{2}$	Fuzzy line.
	867	857 L	$\frac{1}{2}$	
	900 } 913 }	884 } B	$\frac{1}{2}$	
	913 } 917 }	898 } B	1	
	917 } 942 }	902 } B	$\frac{1}{2}$	
	1010 } 1061 }	980 } B	0 to 1	
	1020	987 L	1	Broad.
	1040	1003 L	1	,,
	1061	1020		End of spectrum.
Isobutyric acid (Kahlbaum).	893	872 L	1	
	900 } 913 }	884 } B	$\frac{1}{2}$	
	913 } 917 }	898 } B	1	
	917 } 930 }	902 } B	$\frac{1}{2}$	
	1020	987 L	$\frac{3}{4}$	
	1040	1003 B	..	
	1067	1025 B	..	As in ethyl iodide.
	1085	1037 B	..	,, ,,
	1135	1072 B	..	,, ,,
	1020 } 1075 }	987 } B	$\frac{1}{2}$	
	1075 } 1103 }	1030 } B	$\frac{1}{2}$ to 1	
	1103	1050		End of spectrum.

Compound.	Lines (L) and bands (B).			
	Approximate wave-length.	Scale on plates.	Intensity.	
Valerianic acid (Kahlbaum).	746	745 L	$\frac{1}{2}$	Broad line.
	907	892 B	0 to $\frac{1}{4}$	
	913	898 B		
	913	898 B	$\frac{3}{4}$ to 1	
	917	902 B	1 to $\frac{3}{4}$	
	917	902 B	$\frac{1}{4}$ to 0	
	930	912 B		
	949	927 L	$\frac{1}{2}$	Very fine.
	971	947 L	$\frac{1}{2}$	,,
	1002	972 L	$\frac{1}{2}$	,,
	1027	992 L	$\frac{1}{2}$	,,
	1040	1003 L	$\frac{3}{4}$	Broad line.
	1020	987 B	0 to 1	
	1047	1010 B		
	1047	1010	1	End of spectrum.
Aceto-acetic ether (Hodgkinson).	741	740 L	$\frac{1}{2}$	Broad line.
	886	872 L	$\frac{1}{4}$	
	895	880 B	0 to $\frac{1}{2}$	
	907	892 B		
	907	892 B	$\frac{1}{4}$	
	913	898 B		
	913	898 B	$\frac{3}{4}$	
	917	902 B		
	1003	972 L	$\frac{1}{4}$	
	1003	972 B	0 to 1	
	1133	1070 B		
	1020	987 L	$\frac{1}{4}$	
	1040	1003 L	$\frac{1}{2}$	
	1047	1010 B	$\frac{1}{2}$ to $\frac{3}{4}$	
	1066	1025 B	$\frac{3}{4}$ to 1	
	1096	1045 B	$\frac{1}{2}$ to 1	
	1107	1055 B	1 to $\frac{3}{4}$	
	1132	1070	1	End of spectrum.
Diethyl aceto-acetic ether (Hodgkinson).	900	884 B	0 to $\frac{1}{2}$	
	907	892 B		
	907	892 B	$\frac{1}{4}$	
	913	898 B		
	913	898 B	1	
	917	902 B		
	917	902 B	$\frac{1}{2}$ to 0	
	922	905 B		
	1003	972 L	$\frac{1}{2}$	
	1020	987 L	$\frac{1}{2}$	
	1040	1003 L	$\frac{1}{2}$	Broad line.
	1047	1010 B	$\frac{1}{2}$ to 1	
	1066	1025 B	1 to	
	1020	987 B	0 to $\frac{1}{2}$	Background of spectrum.
	1066	1025 B		
	1066	1025 B	$\frac{1}{2}$	,, ,,
	—	1150 B		
	1096	1045 B	1 to $\frac{1}{2}$	
	1107	1055 B		
	1159	1095 B	1 to $\frac{1}{2}$	
	1171	1105 B		
	—	1160	1	End of spectrum.

Compound.	Lines (L) and bands (B). Approximate wave-length.	Scale on plates.	Intensity.	
Glycerine (PRICE'S).	915	900 L	1	Fine line.
	915, 949	900, 927 B	¼	
	949, 985	927, 959 B	¼ to 0	
	1002	972 L	1	
	1002, 1003	972, 973 B	¼	
	1003	973	1	End of spectrum.
Benzine (ARMSTRONG).	710	713 L	½	Fuzzy line.
	862	852 L	½	
	867	857 L	½	
	867, 872	857, 861 B	¼	
	872, 888	861, 875 B	At 861 to 872, 1; from 872 to 875, 1 to 0	
	915	900 B	0 to ½	
	930	912 B	¼ to 0	
	963, —	940, 1150 B	0 to ½	Shaded gradually as background.
	1002	972 B	1	As in ethyl iodide.
	1020	987 B	1	,, ,,
	1040	1003 B	1	,, ,,
	1066	1025 B	1	,, ,,
	1096, 1105	1045, 1052 B	At 1045, 1; from 1045 to 1052, 1 to 0	
	1137, 1149	1074, 1084 B	At 1074, 1; from 1074 to 1084, 1 to 0	
	—	1125 B	At 1125, 1; from 1125 to 1135, 1 to 0	
	—	1135 B		
	—	1160	1	End of spectrum.
Phenyl bromide (HODGKINSON).	710	713 L	½	Fuzzy line.
	867	857 L	½	
	867, 872	857, 861 B	½	
	872, 885	861, 872 B	1 to 0	As in benzine.
	963, —	940, 1150 B	0 to ½	,, ,,
	1020	987 B	1	,, ,,
	1040	1003 B	1	,, ,,
	1066	1025 B	1	,, ,,
	1096, 1105	1045, 1052 B	1	,, ,,
	1137, 1149	1074, 1084 B	1	,, ,,
	—	1125 B	1	,, ,,
	—	1135 B		
	—	1160	1	End of spectrum.

Compound.	Lines (L) and bands (B).			
	Approximate wave-length.	Scale on plates.	Intensity.	
Benzyl chloride (Hodgkinson).	711	713 L	$\frac{1}{2}$	Fuzzy line.
	867	857 L	$\frac{1}{2}$	
	867	857 B	$\frac{1}{2}$	
	873	861		
	873	861 B	1 to 0	As in benzine.
	885	872		
	964	940 B	0 to $\frac{1}{2}$	" "
	—	1150		
	899	884 B	0 to $\frac{3}{4}$	
	907	892	$\frac{3}{4}$ to 0	
	907	892 B	$\frac{1}{4}$	
	913	898		
	913	898 B	$\frac{1}{2}$	
	918	902		
	918	902 B	$\frac{1}{4}$	
	922	905		
	922	905 B	$\frac{1}{2}$	
	930	912		
	1020	987 B	1	As in benzine.
	1040	1003 B	1	" "
	1066	1025 B	1	" "
	1097	1045 B	1	" "
	1105	1052		
	1138	1074 B	1	" "
	1149	1084		
	—	1125 B	1	" "
	—	1135		
	—	1160	1	End of spectrum.
Aniline (Kahlbaum).	711	713 B	0 to 1	
	720	722	1 to 0	
	778	775 B	0 to $\frac{1}{4}$	
	781	777		
	781	777 B	1	
	800	795		
	800	795 B	$\frac{1}{4}$ to 0	
	805	800		
	867	857 L	$\frac{1}{2}$	
	867	857 B	$\frac{1}{4}$	
	872	860		
	872	860	1 to 0	Keeps its maximum to 817.
	889	875		
	930	912 L	$\frac{1}{2}$	Broad.
	967	942 L	$\frac{1}{2}$	"
	985	959 B	1	0 to 1
	995	968		1 to 0
	1020	987	1	
	1037	1000		
	1037	1000 B	1 to $\frac{3}{4}$	
	1040	1003		
	1040	1003 B	$\frac{3}{4}$	
	1135	1072		
	1135	1072 B	1	As in ethyl iodide.
	1135	1072 B	$\frac{1}{2}$	
	1165	1100		
	1165	1100	1	End of spectrum.

Compound.	Lines (L) and bands (B).			
	Approximate wave-length.	Scale on plates.	Intensity.	
Dimethyl aniline (Hodgkinson).	867	857 L	¾	
	867	857 B	¼	
	871	860		
	871	860 B	1 to 0	Keeps its maximum to 867.
	887	875		
	907	892 B	0 to ¾	Maximum intensity about 898.
	930	912	¾ to 0	
	972	947 B	0 to 1	Maximum intensity about λ 952.
	995	968	1 to 0	
	1040	1003 B	0 to 1	
	1061	1020		
	1061	1020 B	1	
	1075	1030		
	1075	1030 B	½	
	1097	1045		
	1097	1045 B	1 to ½	
	1105	1052		
	1118	1060	½ to 1	
	1132	1070		
	1132	1070		End of spectrum.
Nitro-benzine (Hodgkinson).	710	713 L	¼	Fuzzy line.
	856	845 B	¼	
	866	856		
	867	857 B	½	
	879	867		
	915	900 L	¼	Broad line.
	949	927 L	¼	,, ,,
	985	959 L	¼	,, ,,
	985	959 B	0 to 1	
	1090	1040		
	1002	972	¼	Broad line.
	1040	1003	¾	,, ,,
	1066	1025	1	,, ,,
	1090	1040	1	End of spectrum.
Turpentine (commercial).	880	868 B	0 to ¾	
	897	882		
	903	888	¾	
	910	895		
	913	898	¾	
	918	902		
	925	908	1	
	942	920		
	949	927	1	
	958	935		
	972	947	¼	
	1002	972	1	Broad line.
	1020	987	1	,,
	1040	1003	1	,,
	1002	972 B	0 to 1	
	—	1160		
	1066	1025 B	1	As in the benzine spectrum.
	1100	1047	1	,, ,,
	1145	1080	1	,, ,,
	—	1130	1	,, ,,

Compound.	Lines (L) and bands (B).			
	Approximate wave-length.	Scale on plates.	Intensity.	
Phenyl-propyl alcohol (Hodgkinson).	716	718 L	$\frac{1}{4}$	
	867 }	857 } B	0 to $\frac{1}{4}$	
	873 }	861 }		
	873 }	861 } B	1	
	879 }	867 }		
	879 }	867 } B	$\frac{1}{2}$ to 0	
	887 }	875 }		
	915 }	900 } B	0 to $\frac{1}{4}$	
	921 }	905 }		
	921 }	905 } B	$\frac{1}{4}$ to 1	
	942 }	920 }	1 to $\frac{1}{4}$	
	942 }	920 } B	$\frac{1}{4}$	
	967 }	942 }		
	949	927 L	$\frac{1}{2}$	
	967 }	942 } B	$\frac{1}{4}$ to 0	
	992 }	965 }		
	992 }	965 } B	1 to $\frac{1}{4}$	
	1011 }	980 }		
	1011 }	980 }	$\frac{1}{2}$	
	1040 }	1003 }		
	1040	1003	1	End of spectrum.
Methyl salicylate (Hodgkinson).	710	713	$\frac{1}{2}$	Fuzzy line.
	861 }	851 }	1	
	877 }	865 }		
	892	878 L	$\frac{1}{4}$	
	907 }	892 }	0 to $\frac{1}{4}$	
	925 }	908 }	$\frac{1}{4}$ to 0	
	966 }	942 }	0 to 1	
	1037 }	1000 }		
	1037	1000	1	End of spectrum.
Benzyl-ethyl ether (Hodgkinson).	710	713	$\frac{1}{2}$	Fuzzy.
	866 }	856 } B	0 to $\frac{1}{4}$	
	873 }	861 }		
	873 }	861 } B	1	
	885 }	872 }		
	885 }	872 }	$\frac{1}{4}$ to 0	
	887 }	875 }		
	907 }	892 }	0 to $\frac{3}{4}$	
	918 }	902 }		
	949 }	927 }	0 to $\frac{1}{2}$	
	985 }	959 }	$\frac{1}{2}$ to 0	
	1040	1003 L	1	Broad line.
	1132	1070	1	,,
	1011 }	980 }	0 to 1	Gradual shade.
	— }	1160 }		
	—	1160	1	End of spectrum.

Compound.	Lines (L) and bands (B).			
	Approximate wave-length.	Scale on plates.	Intensity.	
Dibenzyl acetic ether (Hodgkinson).	867	857 L	$\frac{1}{2}$	
	867	857 B	0 to $\frac{1}{4}$	
	873	861		
	873	861 B	1	
	885	872		
	885	872 B	$\frac{1}{4}$ to 0	
	887	875		
	903	888 L	$\frac{1}{2}$	The position of this line not quite certain.
	913	898 B	0 to $\frac{1}{2}$	
	942	920	$\frac{1}{2}$ to 0	
	986	960 B	0 to 1	Background.
	1102	1050		
	1020	987	1	
	1040	1003	1	
	1102	1050	1	End of spectrum.
Ethylic benzoate (Hodgkinson).	710	713 L	$\frac{1}{4}$	
	732	732 L	$\frac{1}{4}$	
	867	857 L	$\frac{1}{2}$	
	867	857	0 to $\frac{1}{4}$	
	875	863		
	875	863	$\frac{3}{4}$	
	885	872		
	885	872	$\frac{1}{4}$ to 0	
	887	875		
	907	892	0 to $\frac{1}{4}$	
	913	898		
	913	898	$\frac{1}{2}$	
	918	902		
	1023	990	0 to 1	
	1061	1020		
	1061	1020	1	End of spectrum.
Olive oil (commercial)	867	857 L	$\frac{1}{2}$	Fine line.
	895	880	0 to 1	
	907	892		
	918	902	1	
	958	935		
	958	935	0 to $\frac{1}{4}$	
	1040	1003		
	1040	1003	1	End of spectrum.
Allyl alcohol (Kahlbaum).	873	861 B	0 to $\frac{3}{4}$	
	877	865	$\frac{3}{4}$ to 0	
	885	872 B	0 to $\frac{3}{4}$	
	900	885	$\frac{3}{4}$ to 0	
	930	912	0 to $\frac{3}{4}$	
	966	942		
	966	942 L	1	
	966	942	$\frac{1}{4}$	
	989	963		
	989	963 B	$\frac{1}{2}$	
	1037	1000		
	1045	1010	1	End of spectrum.

Compound.	Lines (L) and bands (B).			
	Approximate wave-length.	Scale on plates.	Intensity.	
Allyl sulphide (Kahlbaum).	873 }	861 } B	{ 0 to $\frac{3}{4}$	
	877 }	865 }	{ $\frac{3}{4}$ to 0	
	893 }	878 } B	{ 0 to $\frac{3}{4}$	
	900 }	885 }	{ $\frac{3}{4}$ to 0	
	921 }	905 } B	{ 0 to $\frac{3}{4}$	
	936 }	915 }	{ $\frac{3}{4}$ to 0	
	966	942 L	1	
	989	963 L	1	
	1061 }	1020 } B	0 to 1	Gradual shading.
	— }	1160 }		
	1110	1057 L	1	
	—	1140 L	1	
	—	1160	1	End of spectrum.
Anethol (Hodgkinson)	873 }	861 } B	{ 0 to $\frac{3}{4}$	
	880 }	868 }	{ $\frac{3}{4}$ to 0	
	891 }	877 } B	{ 0 to $\frac{3}{4}$	
	905 }	890 }	{ $\frac{3}{4}$ to 0	
	915 }	900 } B	{ 0 to $\frac{3}{4}$	
	921 }	905 }	{ $\frac{3}{4}$ to 0	
	925 }	908 } B	1	
	930 }	912 }		
	1037 }	1000 } B	{ 0 to $\frac{3}{4}$	
	1044 }	1006 }	{ $\frac{3}{4}$ to 0	
	1061 }	1020 } B	$\frac{3}{4}$	
	1119 }	1060 }		
	1124 }	1065 } B	$\frac{1}{2}$	
	— }	1160 }		
	—	1170	1	End of spectrum.
Citraconic anhydride (Hodgkinson).	710	713 L	$\frac{1}{4}$	
	859	840 L	$\frac{1}{2}$	Broad.
	903 }	888 }	{ 0 to $\frac{1}{2}$ }	Very steep edges.
	907 }	892 }	{ $\frac{1}{2}$ to 0 }	
	913 }	898 }	$\frac{1}{2}$	
	918 }	902 }		
	949	927 L	$\frac{1}{2}$	Broad.
	985	959	$\frac{1}{2}$	,,
	1020	987	$\frac{1}{2}$	,,
	1040 }	1003 } B	{ 0 to $\frac{1}{2}$	
	1080 }	1033 }	{ $\frac{1}{2}$ to 0	
	1105 }	1052 } B	{ 0 to $\frac{1}{2}$	
	1157 }	1092 }	{ $\frac{1}{2}$ to 0	
	1040 }	1003 } B	$\frac{1}{2}$	
	— }	1120 }		
	—	1120 } B	$\frac{1}{2}$ to 1	
	—	1130 }		
	—	1130	1	End of spectrum.

Compound.	Lines (L) and bands (B).			
	Approximate wave-length.	Scale on plates.	Intensity.	
Chloroform (commercial).	720	722 L	$\frac{1}{2}$	
	732	732 L	$\frac{1}{2}$	
	774	772 L	$\frac{1}{2}$	
	792	788 L	$\frac{1}{2}$	
	813	808 L	$\frac{1}{2}$	
	833	824 L	$\frac{1}{2}$	
	846	837 L	$\frac{1}{4}$	
	867	857 L	$\frac{1}{4}$	
	892	878 L	1	Broad and sharp at edges.
	907	892	$\frac{1}{2}$	
	930	912 L	$\frac{1}{2}$	
	949	927 L	$\frac{1}{2}$	
	985	959 L	$\frac{1}{2}$	
	1011	980	$\frac{1}{2}$	
	1066	1025	1	Broad and sharp at edges.
	1165	1100	1	" "
	—	1180	1	" "
	—	1200	1	End of spectrum.
Hydrochloric acid, 6 inches (Hopkin and Williams).	732	732 L	$\frac{1}{4}$	These lines are very faint and would escape notice under ordinary circumstances.
	741	740 L	$\frac{1}{4}$	
	845	836 L		
	867	857 L	$\frac{1}{4}$	
	949	927 L	$\frac{1}{4}$	
Ammonia, ·880 (Hopkin and Williams).	732	732 L	$\frac{1}{2}$	Broad line with nucleus.
	774	772 L	$\frac{1}{2}$	" "
	796	792 B	1	
	799	795 B		
	885	872 L	$\frac{1}{2}$	
	895	880 B	0 to $\frac{1}{4}$	
	903	888 B		
	903	888 B	1	
	907	892 B		
	907	892	$\frac{1}{4}$ to 0	
	913	898		
	915	900 L	1	Fuzzy joining previous band, but having a distinct nucleus.
	934	915 B	0 to 1	
	949	927 B	1 to $\frac{1}{2}$	
	949	927 B	$\frac{1}{2}$	
	999	970 B		
	972	947 B	$\frac{1}{2}$ to 1	
	985	959 B	1 to $\frac{1}{2}$	
	999	970		
	1011	980	$\frac{1}{2}$ to 1	
	1011	980	..	End of spectrum.

Compound.	Lines (L) and bands (B).			
	Approximate wave-length.	Scale on plates.	Intensity.	
Nitric acid, 1·360 (Hopkin and Williams).	774	772 L	$\frac{1}{2}$	Broad line with nucleus
	845	836 L	$\frac{1}{2}$	" "
	895 }	880 } B	$\frac{1}{4}$	
	942 }	920 }		
	942 }	920 } B	0 to 1	
	949 }	927 }	1 to $\frac{1}{2}$	
	949 }	927 } B	$\frac{1}{2}$	Background.
	1011 }	980 }		
	985 }	959 } B	$\frac{1}{2}$ to 1	
	1002 }	972 }	1 to $\frac{1}{2}$	
	1011 }	980 }		
	1037 }	1000 }	$\frac{1}{2}$ to 1	
	1037	1000	1	End of spectrum.
Sulphuric acid.	799	795 L	$\frac{1}{4}$	
	813	808 L	$\frac{1}{4}$	
	867	857	$\frac{1}{4}$	
	882 }	870 }	0 to 1	
	975 }	950 }		
	975	950	..	End of spectrum.
Water, 6 inches.	732 }	732 } B	$\frac{1}{4}$	Traces of a line at both 732, 824.
	833 }	824 }		
	833 }	824 } B	Gradual shading from $\frac{1}{2}$ to $\frac{3}{4}$	Traces of a line at 920.
	942 }	920 }		
	942 }	920 } B	1	Traces of a line at 959.
	984 }	958 }		
	984 }	958 }	$\frac{3}{4}$	
	1000 }	970 }		
	1000 }	970 }	1	The central part shows line 972.
	1004 }	974 }		
	1004 }	974 }	$\frac{3}{4}$	
	1045 }	1010 }		1010 end of spectrum.
Water, 2 feet.	732	732 L	$\frac{3}{4}$	
	732 }	732 }	Band shading from $\frac{1}{2}$ to $\frac{1}{4}$	
	833 }	824 }		
	833	824 L	$\frac{3}{4}$	

Compound.	Lines (L) and bands (B).			
	Approximate wave-length.	Scale on plates.	Intensity.	
¾ alcohol, ¼ benzine.	710	713 L	¼	
	741	740 L	¼	
	867	857 L	¼	
	873	861 B	½	
	885	872 B	½	
	907	892 B	½	
	921	905 B	0 to 1	
	930	912 L	¼	
	958	935 L	½	Broad line.
	985	959 B	½	Trace of line at 959.
	1105	1052 B	½	
	1002	972 L	¼	
	1020	987 L	¼	
	1037	1000 B	¼ to ¾	
	1044	1008 B	¼ to ¾	
	1075	1030 B	¼ to ¾	
	1090	1040 B	¼ to ¾	
	1097	1045 B	¼ to ¾	
	1105	1052 B	¼ to ½	
	1105	1052 B	½ to 1	
	1124	1065 B		
	1124	1065	1	End of spectrum.

INDEX

TO THE

PHILOSOPHICAL TRANSACTIONS

FOR THE YEAR 1881.

FRANKLIN INSTITUTE LIBRARY

A.

ABEL's *theorem*, 713, 751.
ABNEY (W. DE W.) and FESTING (E. R.). On the Influence of the Atomic Grouping in the Molecules of Organic Bodies on their Absorption in the Infra-red Region of the Spectrum, 887.
Absorption in the infra-red region of the spectrum, 887.

B.

BAKER (E. C.). Researches on the Minute Structure of the Thyroid Gland, 577 (for contents see p. 577).
Batrachia, structure and development of the skull in the, 1 (see PARKER).
BESSEL's *functions in connexion with* RICCATI's *equation*, 762, 819.
BOTTOMLEY (J. T.). On the Thermal Conductivity of Water, 537.

C.

Calamostachys Binneyana, 298.
CAYLEY (A.). Addition to Mr. ROWE's Memoir on ABEL's Theorem, 751.
Coal-Measures, fossil plants of, 283 (see WILLIAMSON).
Cochlea of Ornithorhynchus, compared with that of Mammals and Birds, 267.
Conductivity, thermal, *of water*, 537.
CROOKES (W.). On the Viscosity of Gases at High Exhaustions, 387 (for contents see p. 387).
Cubic curve in space, on the forty-eight coordinates of, 375.
Curvilinear coordinates, conjugate, *in three dimensions*, 611.

D.

DARWIN (G. H.). On the Tidal Friction of a Planet attended by several Satellites, and on the Evolution of the Solar System, 491 (for contents see p. 491).
Definite integrals in connexion with RICCATI'S *equation*, 759 (see GLAISHER).
Dinosaur, a large, *from the Wealden formation*, 653.

E.

Echinodermata, locomotor system of, 829 (see ROMANES).
Electric currents, induction of, 307.
Electric discharge, resistance of air to, at different degrees of exhaustion, 406.
Electricity, distribution of, *on a tore*, 642.
Electrostatic capacity of glass, 355; of liquids, 369.
EWART (J. C.) and ROMANES (G. J.) (see ROMANES).

F.

FESTING (E. R.) and ABNEY (W. DE W.) (see ABNEY).
Films, colours of, 451; electrical resistance of, 457.
Fluid motion around a tore, 644.
Fossil plants of coal-measures, 283 (see WILLIAMSON).

G.

GLAISHER (J. W. L.). On RICCATI'S Equation and its Transformations, and on some Definite Integrals which satisfy them, 759.—Introduction, 759 (for the headings of the eight sections of the paper see p. 766).

H.

HICKS (W. M.). On Toroidal Functions, 609.—General theory of conjugate curvilinear coordinates in three dimensions. Special case, 611; zonal toroidal functions, 617; sectorial and tesseral functions, 629; tores with no central opening, 638; examples and applications, 640.
HOPKINSON (J.). Electrostatic Capacity of Glass, II., and of Liquids, 355.
HULKE (J. W.). Polacanthus Foxii, a large undescribed Dinosaur from the Wealden formation in the Isle of Wight, 653.

I.

Induction balance, five-plate, examination of, 366.
Induction of electric currents, 307.

L.

LANGLEY (J. N.). On the Histology and Physiology of Pepsin-forming Glands, 663 (for contents see p. 663).
LOCKYER (J. N.). Researches in Spectrum-Analysis in connexion with the Spectrum of the Sun.—No. V., 561 (for contents see p. 561).

M.

MAXWELL'S *law connecting specific inductive capacity and refractive index*, comparison of, with experimental results, 373.
MAXWELL'S *law of constancy of coefficient of viscosity of gases when pressure changes*, 431, 441.
Megalania prisca, 547.
Molecular grouping, elicited from absorption in the infra-red, 887.

N.

NEWTON'S *Table of Colours*, revision of, 451.
NIVEN (C.). On the Induction of Electric Currents in Infinite Plates and Spherical Shells, 307.

O.

Ornithorhynchus, cochlea of, compared with that of Mammals and Birds, 267.
OWEN (R.). Description of some Remains of the Gigantic Land-Lizard (Megalania prisca, OWEN) from Australia.—Part III., 547.

P.

PARKER (W. K.). On the Structure and Development of the Skull in the Batrachia.—Part III., 1.
Pepsin-forming glands, 663 (see LANGLEY).
Polacanthus Foxii, 653.
PRITCHARD (U.). The Cochlea of the Ornithorhynchus platypus compared with that of ordinary Mammals and of Birds, 267.

R.

Refractive indices of various liquids, 373.
REINOLD (A. W.) and RÜCKER (A. W.). On the Electrical Resistance of Thin Liquid Films, with a Revision of NEWTON'S Table of Colours, 447.
Repulsion arising from radiation, in air and gases, 403, 411, 415, 418, 420, 424.
Resistance, electrical, *of a liquid film*, 463; of a liquid in mass, 470; relation of, to temperature and constitution of a liquid, 475.
RICCATI'S *equation and its transformations*, 759 (see GLAISHER).
ROMANES (G. J.) and EWART (J. C.). Observations on the Locomotor System of Echinodermata.—Part I. Morphology, 829; ambulacral system, 829; general homologies, 834; nervous system of Echinus, 835. Part II. Physiology, 836; natural movements, 836; stimulation, 848; section, 856; general summary, 869; literature, 877; description of the figures, 882.
ROWE (R. C.). Memoir on ABEL'S Theorem, 713.
RÜCKER (A. W.) and REINOLD (A. W.) (see REINOLD).

S.

Skull, structure and development of, *in the Batrachia*, 1.
Solar system, evolution of, 512.
Spectra of gases at high exhaustions, 414, 418, 420, 426.

Spectrum-analysis in connexion with the sun, 561 (see LOCKYER).
Spherical shell, induction of electric current in, 345.
SPOTTISWOODE (W.). On the Forty-eight Coordinates of a Cubic Curve in Space, 375.
STOKES (G. G.). Note on the Reduction of Mr. CROOKES's Experiments on the Decrement of the Arc of Vibration of a Mica Plate oscillating within a Bulb containing more or less rarefied gas, 435.
Sun, spectrum of the, 561 (see LOCKYER).

T.

Tidal friction of a planet attended by several satellites, 491.
Thyroid gland, 577 (see BABER).
Toroidal functions, 609 (see HICKS).

V.

Viscosity of gases, 387, 435.

W.

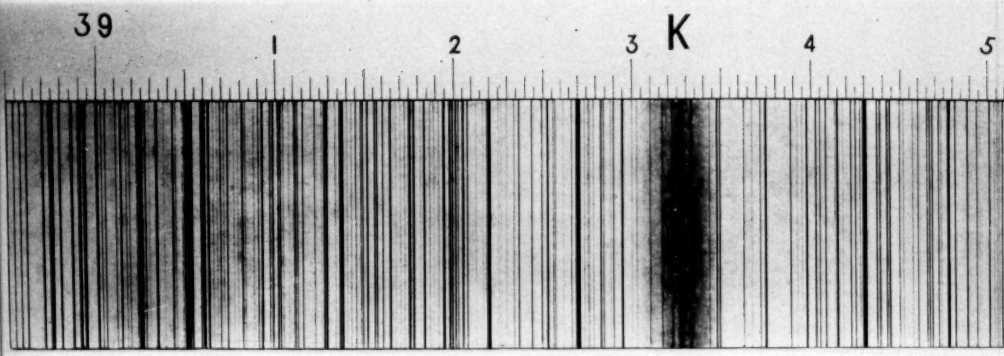
Water, thermal conductivity of, 537.
Wave-lengths, of metallic and solar lines, *tables of*, 573.
Wealden formation, description of a large Dinosaur from the, 653.
WILLIAMSON (W. C.). On the Organization of the Fossil Plants of the Coal-Measures.—Part XI., 283.—General conclusions respecting the Carboniferous Lycopods, 295; Calamostachys Binneyana, 298; index to the plates, 302.

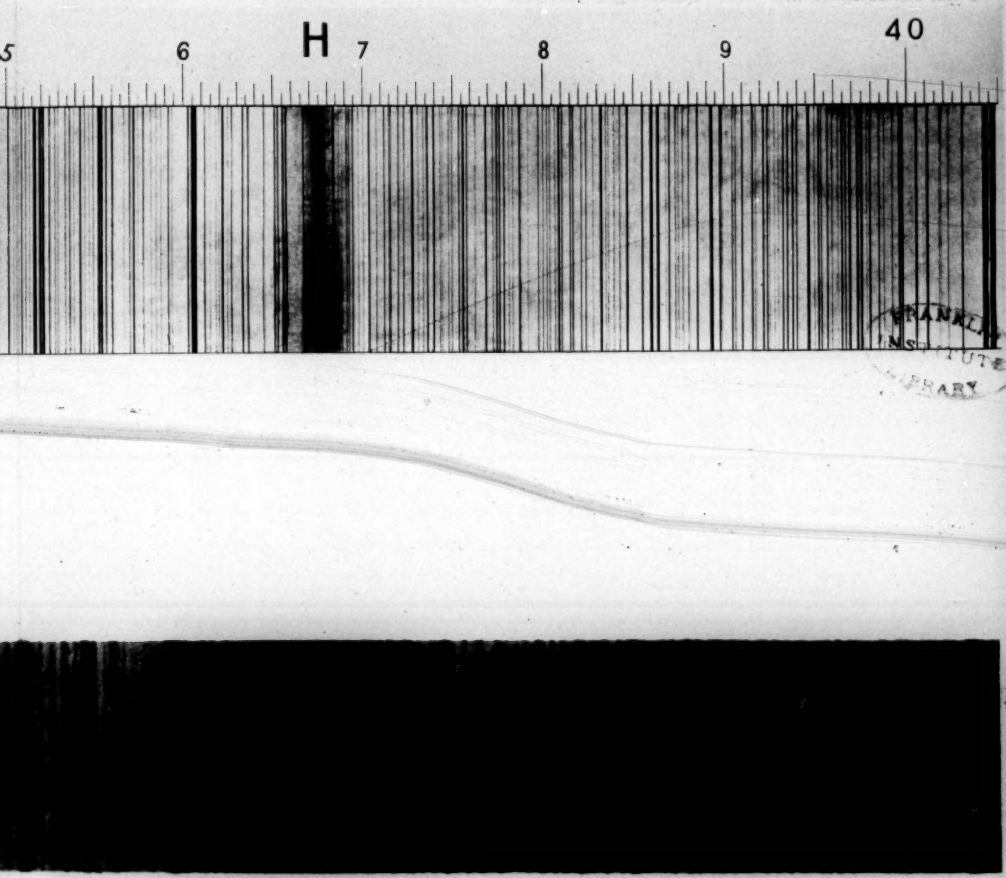
Z.

Zymogen, 696.

LONDON:
PRINTED BY HARRISON AND SONS, ST. MARTIN'S LANE.

Lockyer.





FRANKLIN INSTITUTE LIBRARY

Fig. 1

Fig. 2

Fig. 3

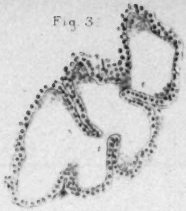


Fig. 5

Fig. 4

FRANKLIN INSTITUTE LIBRARY

Fig. 9.

Fig. 7.

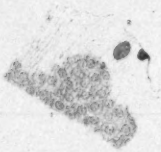


Fig. 8.

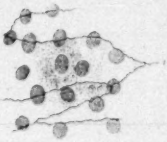


Fig. 6.

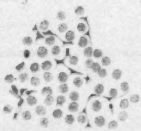


Fig. 12.

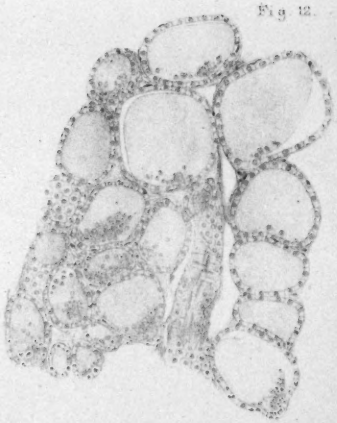


Fig. 11.

Fig. 10.

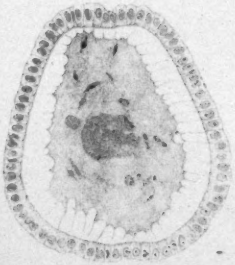


Fig. 13

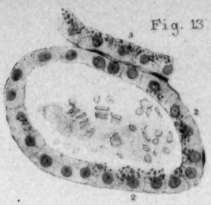
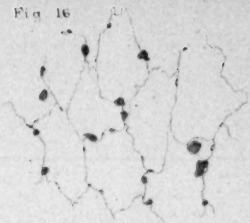


Fig. 14

Fig. 16



FRANKLIN INSTITUTE LIBRARY

Fig. 15

Fig. 17.

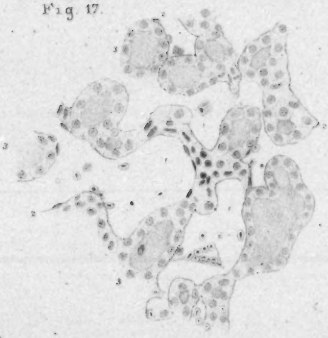


Fig. 20.

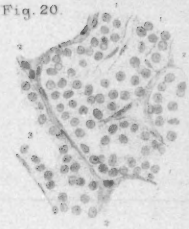
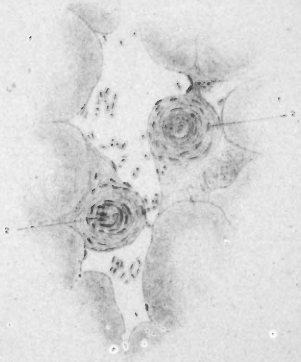


Fig. 19

Fig. 18



E.C. Baber del.

West Newman & Co. lith.

Fig. 1.

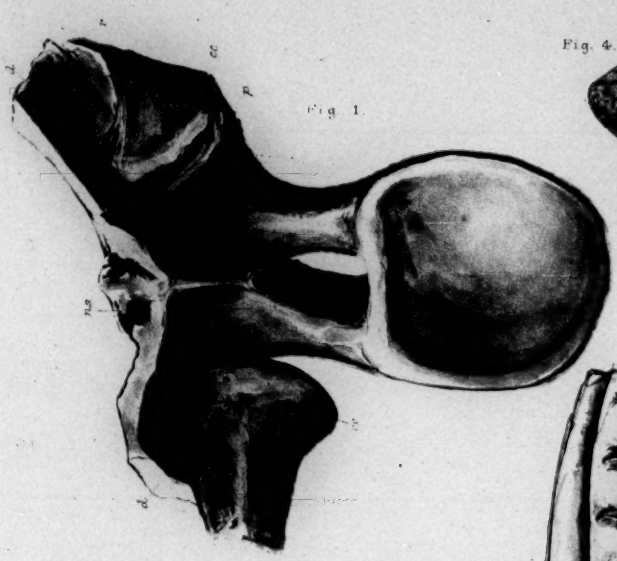
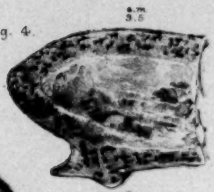
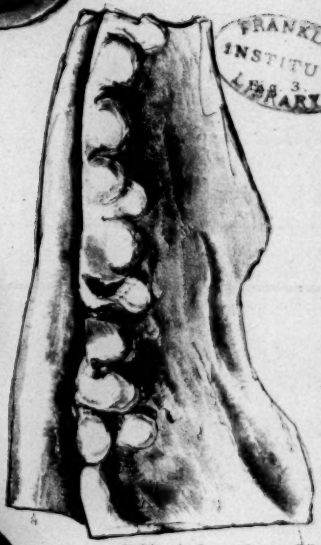


Fig. 4.



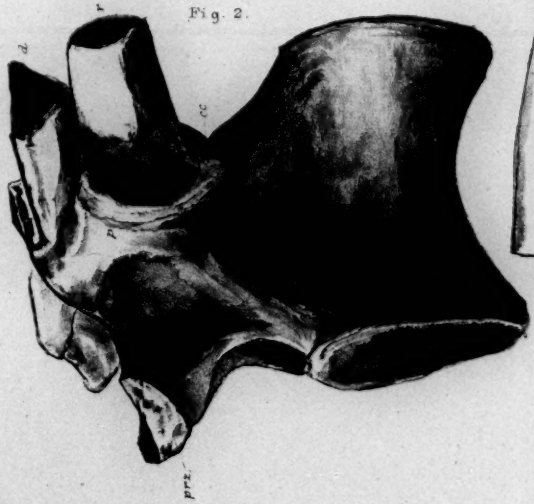
Cross Section at d.
Slightly reduced.

FRANKLIN INSTITUTE LIBRARY



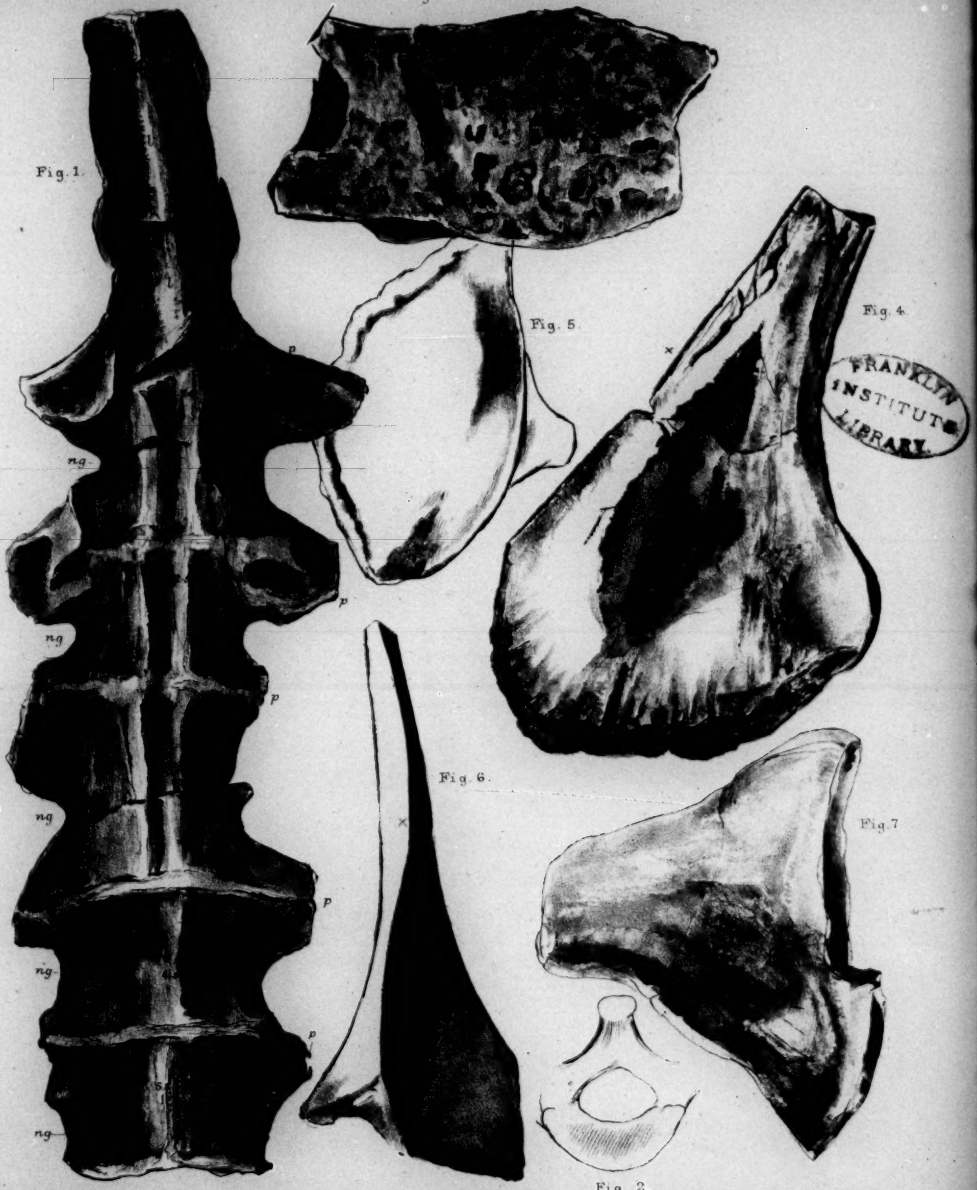
Slightly reduced. Beginning Keil.

Fig. 2.



J. W. Hulke del.
W. H. Wesley lith.

West Newman & Co. imp.



FRANKLIN INSTITUTE LIBRARY

J. W. Hulke del.
W. H. Worley lith.

West, Newman & Co. imp.

Fig. 1.

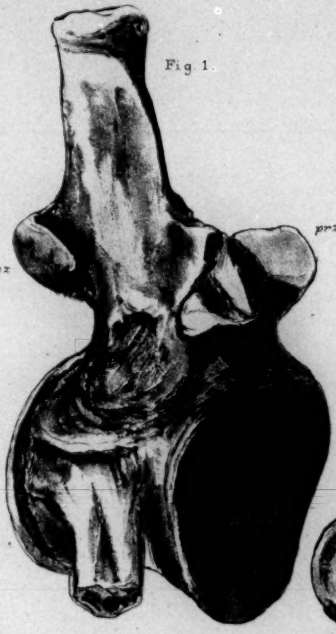


Fig. 2.

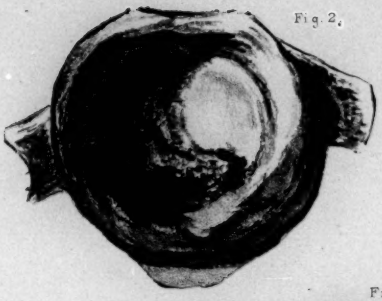
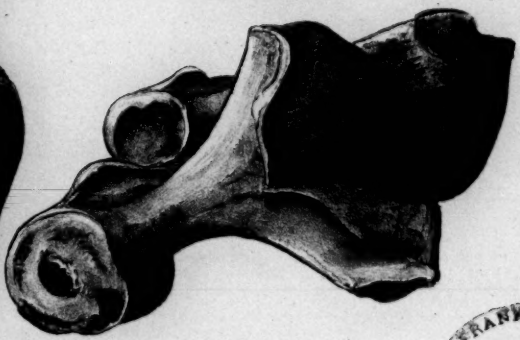
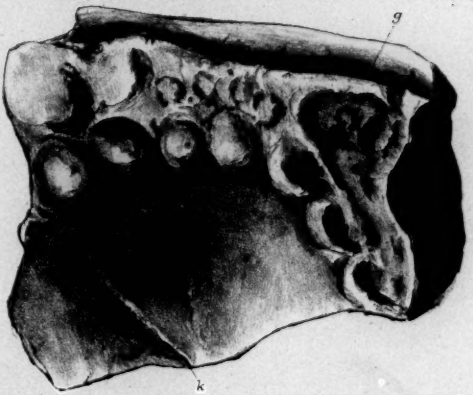


Fig. 3.



FRANKLIN INSTITUTE LIBRARY.

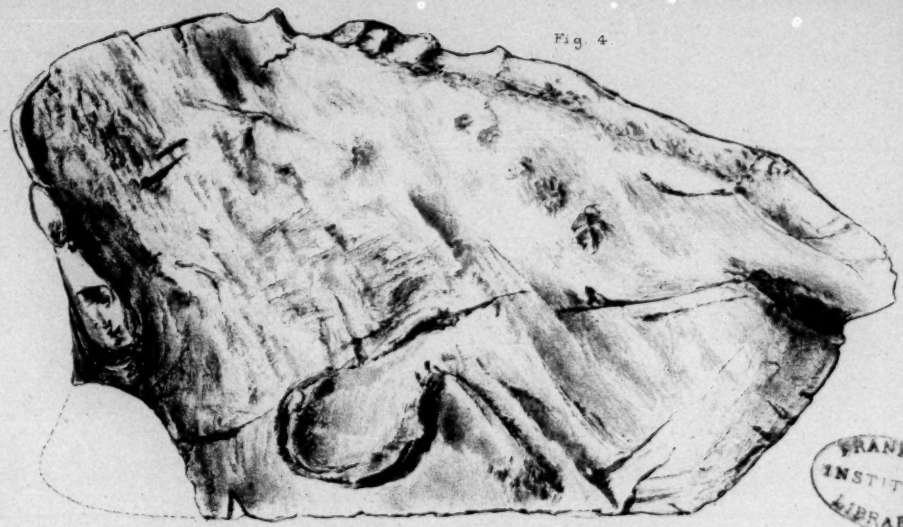
Fig. 4.



J.W.Hulke del.
W.H.Worley lith.

West, Newman & Co.ª imp.

Fig. 4



FRANKLIN INSTITUTE LIBRARY

Fig. 3.

Fig. 1.

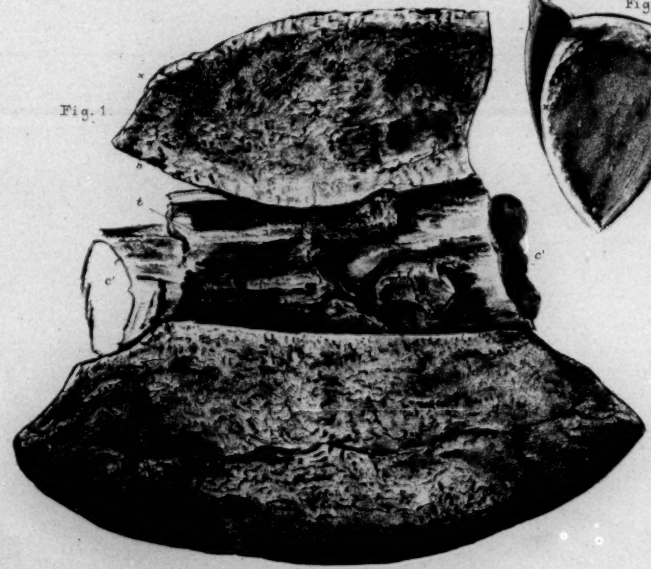


Fig. 2

J.W.Hulke del.
W.H.Wesley lith.

West, Newman, & Co. imp.

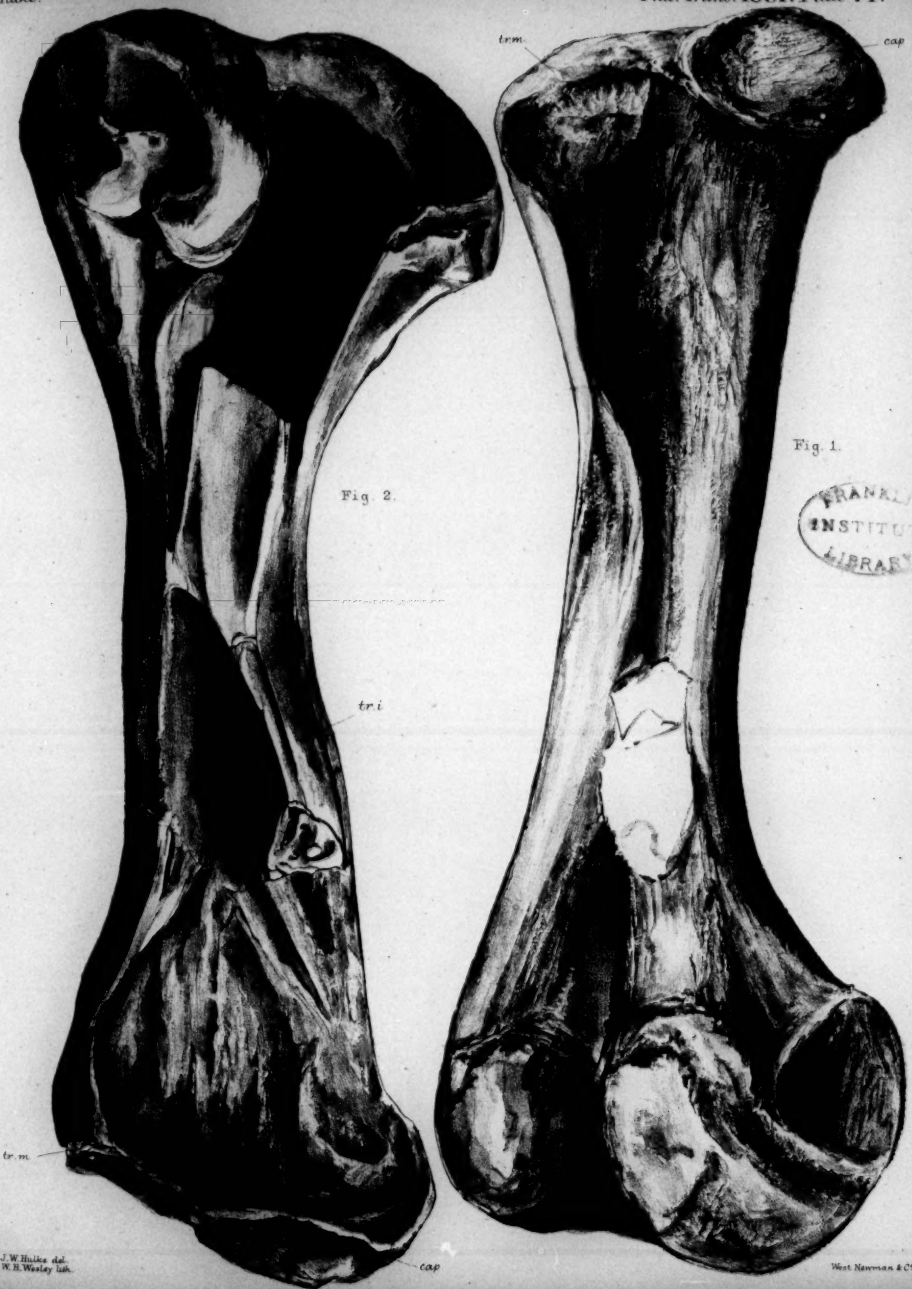


Fig. 2.

Fig. 1.

FRANKLIN INSTITUTE LIBRARY

J.W.Hulke del.
W.B.Wesley lith.
West, Newman & Co. imp

Fig. 2.

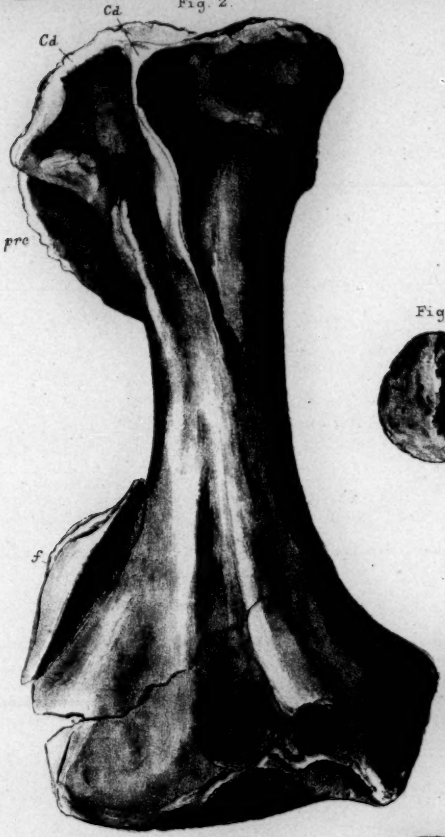


Fig. 1.

Fig. 6.

FRANKLIN INSTITUTE LIBRARY

Fig. 4.

Fig. 3.

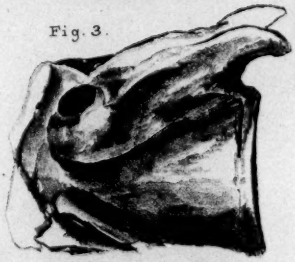
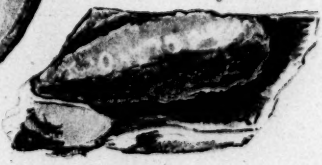


Fig. 5.



J.W. Hulke del.
W.H. Wesley lith.

West, Newman & Co. imp.

Fig. 2.

Fig. 1

FRANKLIN INSTITUTE LIBRARY

J.W. Hulke del.
W.H. Wesley lith.

West, Newman & Co. imp.

1. × 360

2. × 250.

3. × 370.

4. × 410.

5. × 260.

6(a) × 330.

6(b) × 410.

7. × 350.

8. × 340.

9. × 410.

10. × 580.

11. × 370.

FRANKLIN INSTITUTE LIBRARY

West, Newman & Co. imp.

1. × 640

2. × 420

FRANKLIN INSTITUTE LIBRARY

4. × 420

3. × 420

5. × 440

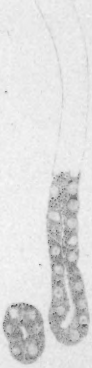
6. × 320

7. × 280

9. × 380

8. × 380

10. × 420



11. × 260

West, Newman & Co. imp.

Fig. 1.

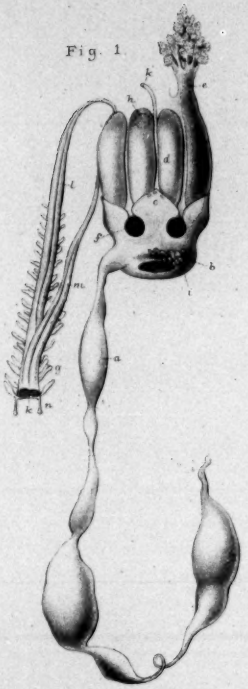
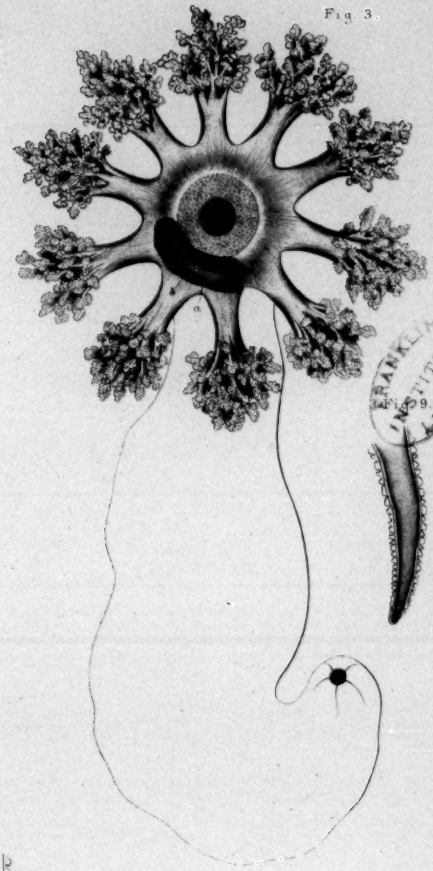


Fig. 6

Fig. 3.



FRANKLIN INSTITUTE LIBRARY

Fig. 9.

Fig. 7.

Fig. 2.

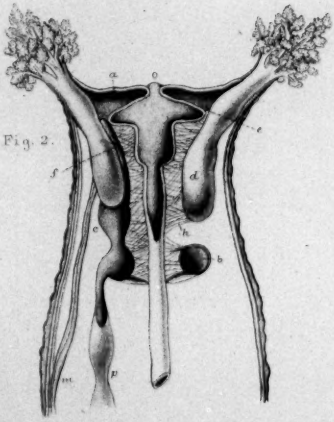


Fig. 8.

Fig. 4.

Fig. 5.

J.C. Ewart. del.

West, Newman & Co. lith.

Fig. 10.

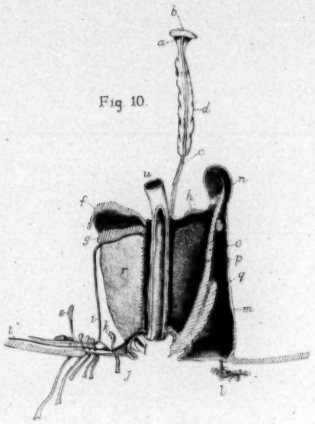
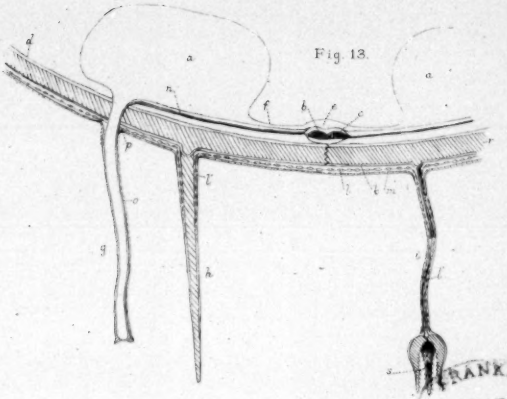


Fig. 13.



FRANKLIN INSTITUTE LIBRARY

Fig. 11.

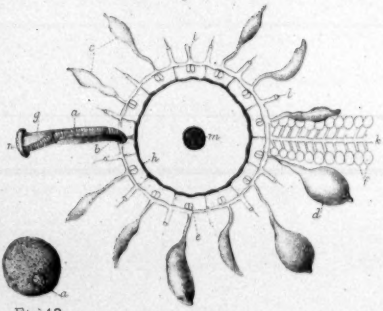


Fig. 12.

Fig. 14.

Fig. 16.

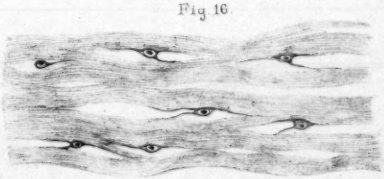


Fig. 15.

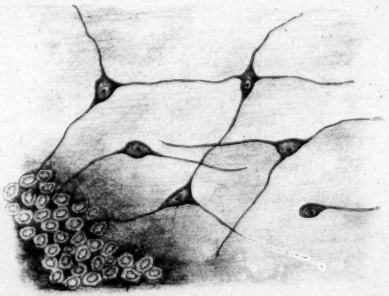


Fig. 18.

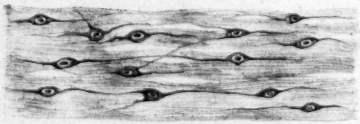


Fig. 17.

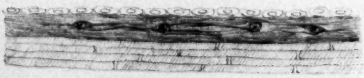
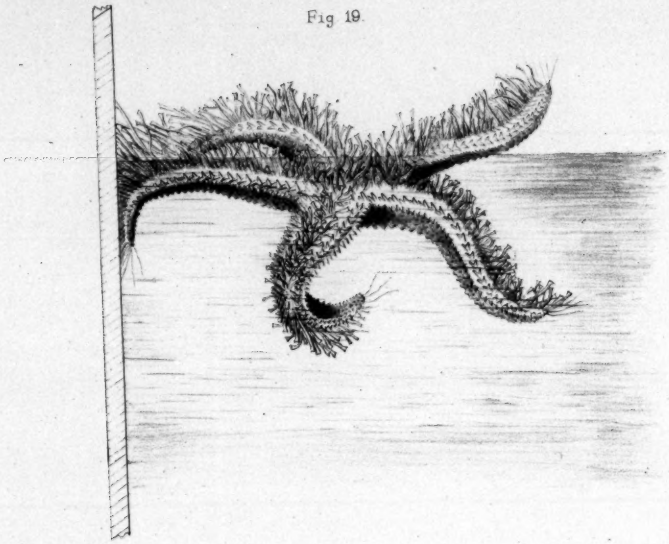


Fig. 19.



FRANKLIN INSTITUTE LIBRARY

Fig. 20.

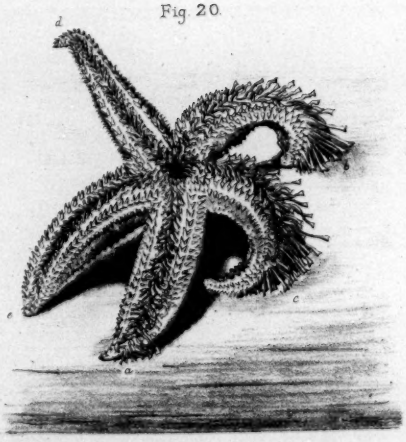
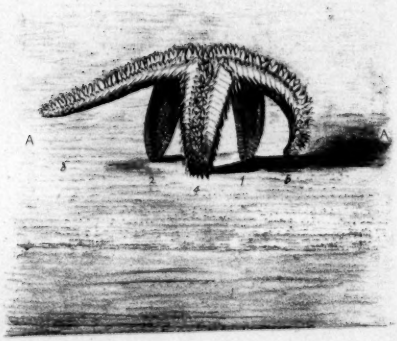


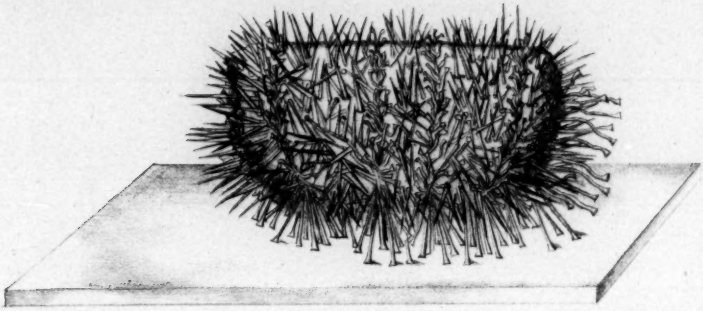
Fig. 21.



J.C.Ewart del.

West, Newman & C° lith.

Fig. 23.



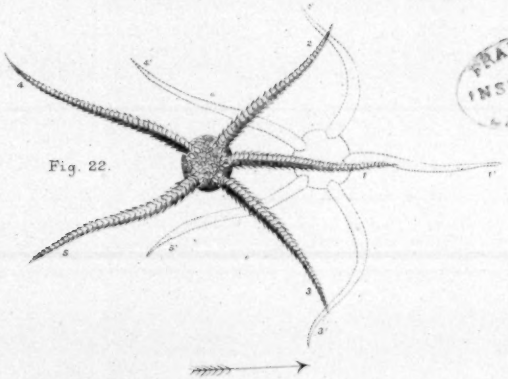


Fig. 22.

FRANKLIN INSTITUTE LIBRARY

Fig. 24.

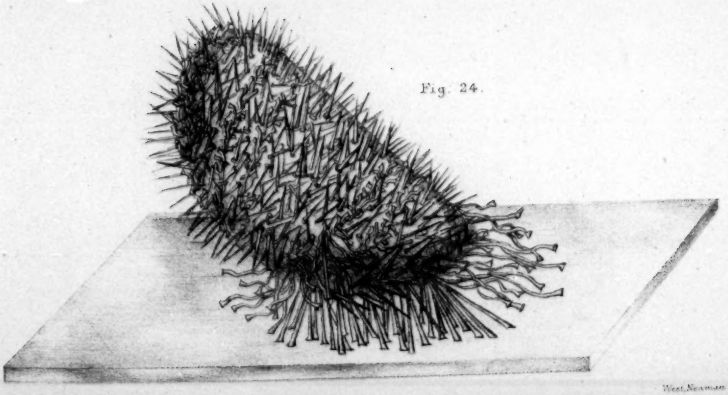


Fig. 25

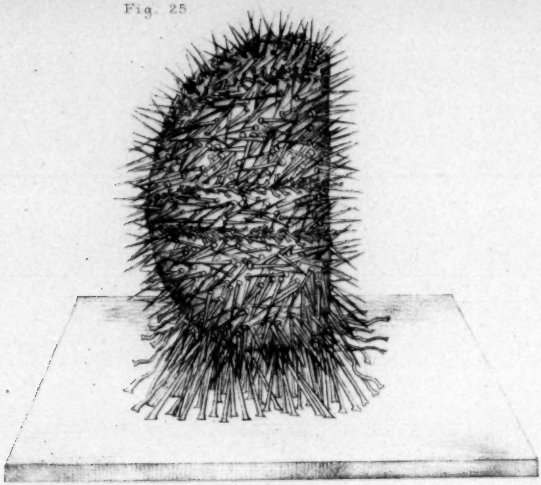
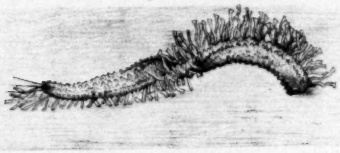


Fig. 27.



FRANKLIN INSTITUTE LIBRARY

Fig. 26

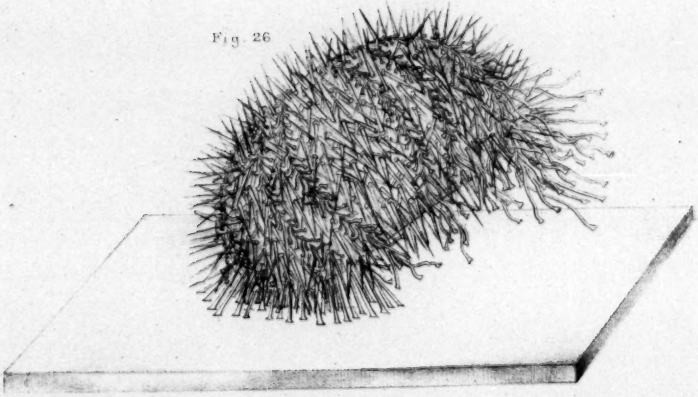


Fig. 28.

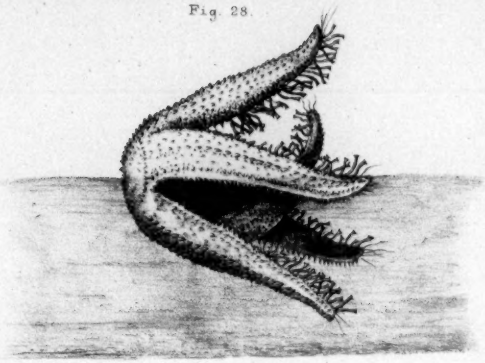
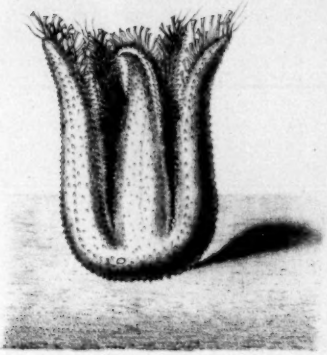


Fig. 29.



FRANKLIN INSTITUTE LIBRARY

Fig. 30.

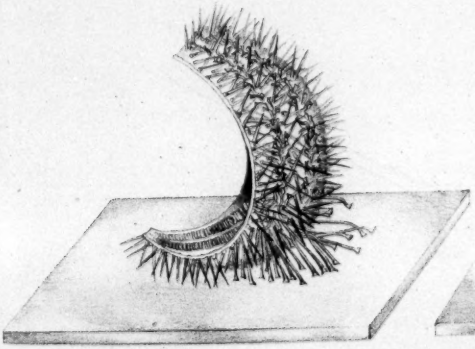


Fig. 31.

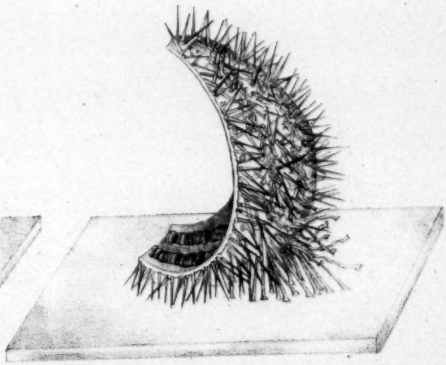
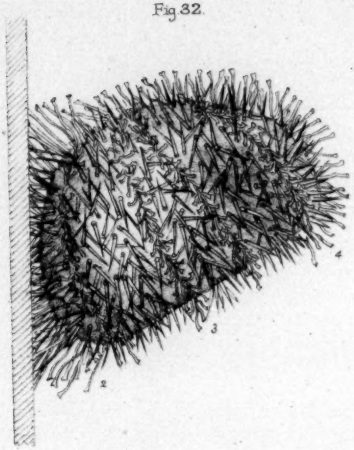


Fig. 32.



FRANKLIN INSTITUTE LIBRARY

Fig. 33.

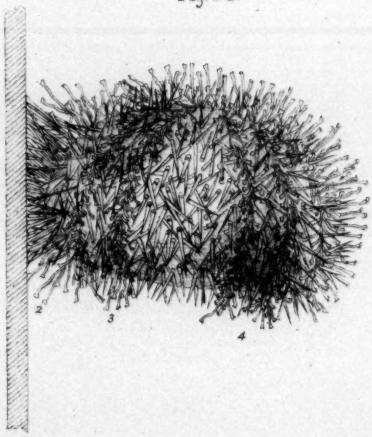
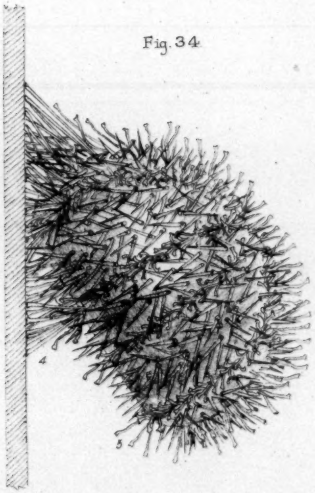
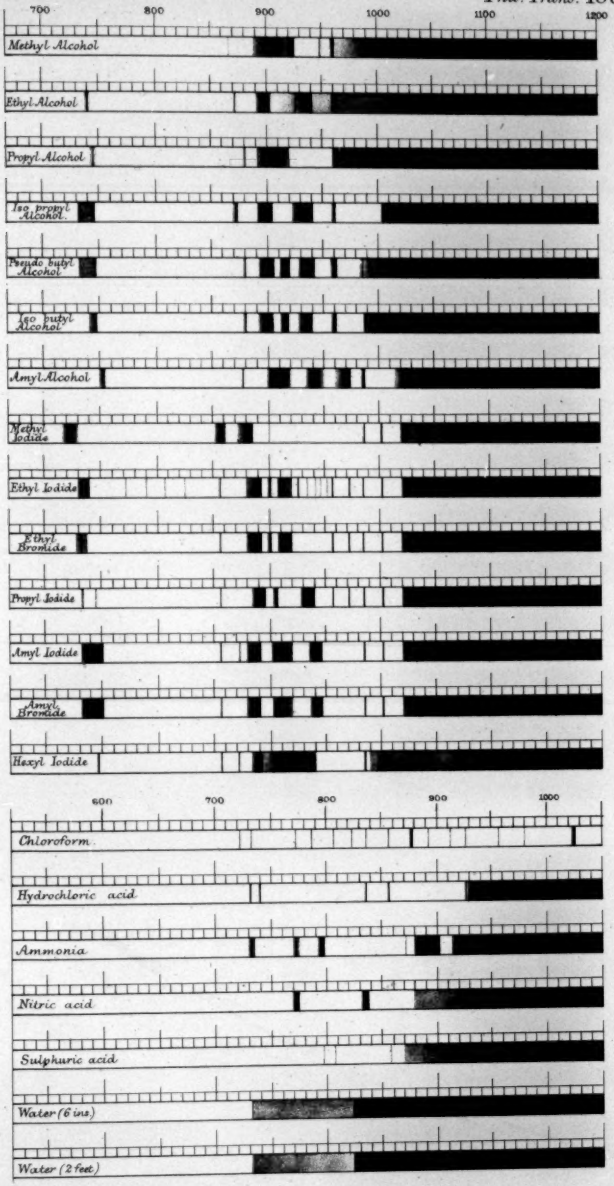


Fig. 34.



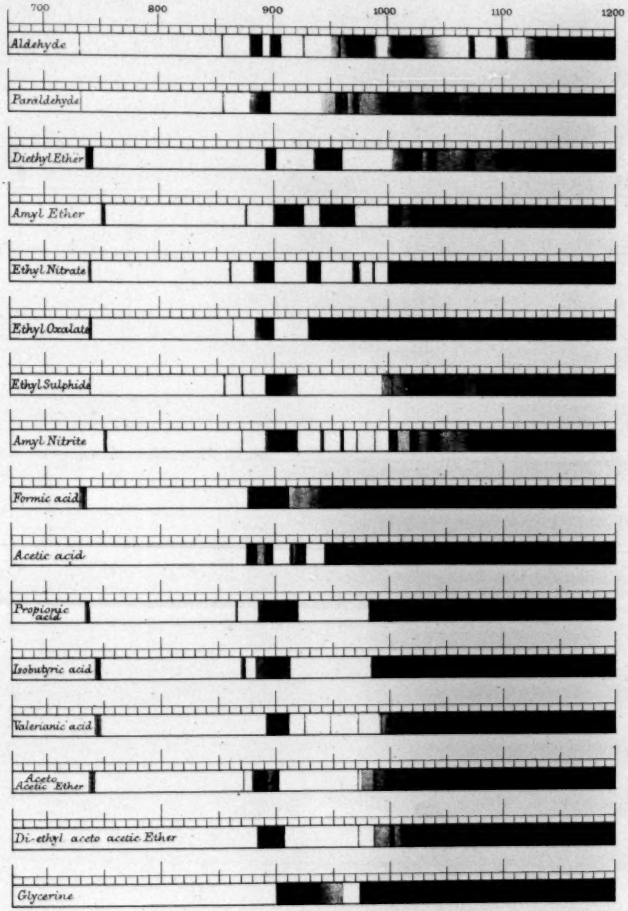
J. C. Ewart del.

West, Newman & Co. lith.



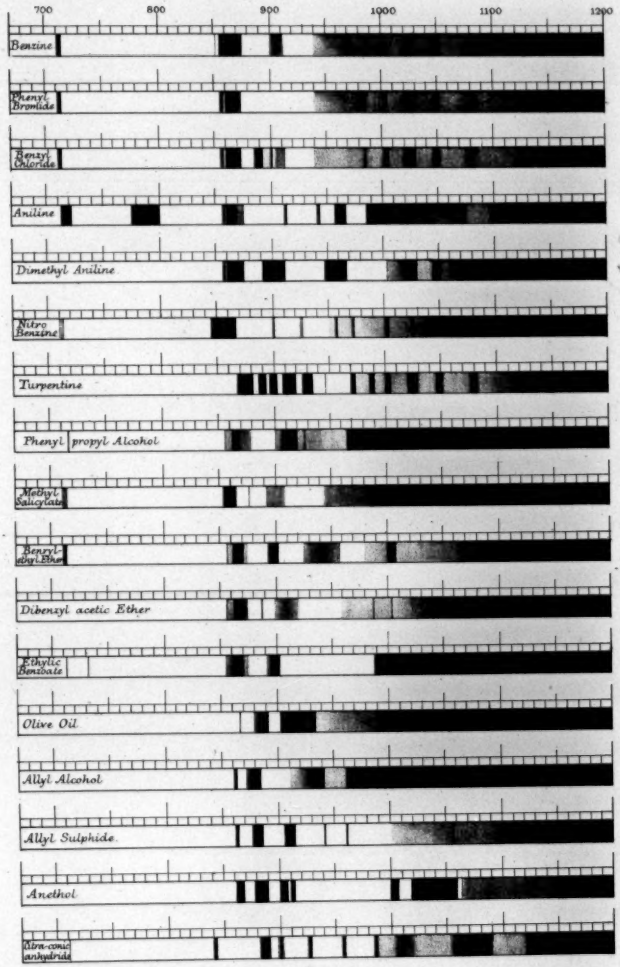
FRANKLIN INSTITUTE LIBRARY

West, Newman & Co. lith.



FRANKLIN INSTITUTE LIBRARY

West, Newman & Co. imp.



FRANKLIN INSTITUTE LIBRARY

W. Rhan lith. West, Newman & Co.t imp.